An analytic approach for obtaining maximal entropy OWA operator weights *

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Abstract

One important issue in the theory of Ordered Weighted Averaging (OWA) operators is the determination of the associated weights. One of the first approaches, suggested by O’Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness; algorithmically it is based on the solution of a constrained optimization problem. In this paper, using the method of Lagrange multipliers, we shall solve this constrained optimization problem analytically and derive a polynomial equation which is then solved to determine the optimal weighting vector.

Keywords: OWA operator, dispersion, degree of orness

1 Introduction

An OWA operator of dimension $n$ is a mapping $F: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = (w_1, \ldots, w_n)^T$ of having the properties

$$w_1 + \cdots + w_n = 1, \ 0 \leq w_i \leq 1, \ i = 1, \ldots, n,$$

and such that

$$F(a_1, \ldots, a_n) = \sum_{i=1}^n w_i b_i,$$

where $b_j$ is the $j$th largest element of the collection of the aggregated objects $\{a_1, \ldots, a_n\}$.

In [3], Yager introduced two characterizing measures associated with the weighting vector $W$ of an OWA operator. The first one, the measure of orness of the aggregation, is defined as

\[
\text{orness}(W) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i)w_i.
\]

and it characterizes the degree to which the aggregation is like an or operation. It is clear that $\text{orness}(W) \in [0, 1]$ holds for any weighting vector.

The second one, the measure of dispersion of the aggregation, is defined as

\[
\text{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i
\]

and it measures the degree to which $W$ takes into account all information in the aggregation.

It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weighting vector. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation [3, 4], exponential smoothing [6] and learning [5].

Another approach, suggested by O’Hagan [2], determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness. This approach is based on the solution of the following mathematical programming problem:

\[
\begin{align*}
\text{maximize} \quad & -\sum_{i=1}^{n} w_i \ln w_i \\
\text{subject to} \quad & \frac{1}{n-1} \sum_{i=1}^{n} (n-i)w_i = \alpha, \quad 0 \leq \alpha \leq 1 \\
& \sum_{i=1}^{n} w_i = 1, \quad 0 \leq w_i \leq 1, \quad i = 1, \ldots, n.
\end{align*}
\]

Using the method of Lagrange multipliers we shall transfer problem (1) to a polynomial equation which is then solved to determine the optimal weighting vector.
2 Obtaining maximal entropy weights

First we note that \( \text{disp}(W) \) is meaningful if \( w_i > 0 \) and by letting \( w_i \ln w_i \) to zero if \( w_i = 0 \), problem (1) turns into

\[
\text{disp}(W) \rightarrow \max; \quad \text{subject to } \{ \text{orness}(W) = \alpha, \ w_1 + \cdots + w_n = 1, \ 0 \leq \alpha \leq 1 \}.
\]

If \( n = 2 \) then from \( \text{orness}(w_1, w_2) = \alpha\) we get \( w_1 = \alpha \) and \( w_2 = 1 - \alpha \). Furthermore, if \( \alpha = 0 \) or \( \alpha = 1 \) then the associated weighting vectors are uniquely defined as \((0, 0, \ldots, 0, 1)^T\) and \((1, 0, \ldots, 0, 0)^T\), respectively, with value of dispersion zero.

Suppose now that \( n \geq 3 \) and \( 0 < \alpha < 1 \). Let us denote the Lagrange function of constrained optimization problem (1), where \( \lambda_1 \) and \( \lambda_2 \) are real numbers. Then the partial derivatives of \( L \) are computed as

\[
\frac{\partial L}{\partial w_j} = -\ln w_j - 1 + \lambda_1 + \frac{n - j}{n - 1} \lambda_2 = 0, \quad \forall j
\]

(2)

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{n} w_i - 1 = 0
\]

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{n - i}{n - 1} w_i - \alpha = 0.
\]

For \( j = n \) equation (2) turns into

\[-\ln w_n - 1 + \lambda_1 = 0 \iff \lambda_1 = \ln w_n + 1,\]

and for \( j = 1 \) we get

\[-\ln w_1 - 1 + \lambda_1 + \lambda_2 = 0,\]

and, therefore,

\[\lambda_2 = \ln w_1 + 1 - \lambda_1 = \ln w_1 + 1 - \ln w_n - 1 = \ln w_1 - \ln w_n.\]

For \( 1 \leq j \leq n \) we find

\[\ln w_j = \frac{j - 1}{n - 1} \ln w_n + \frac{n - j}{n - 1} \ln w_1 \Rightarrow w_j = \sqrt[n-j]{{w_1^{n-j} w_n^{j-1}}}.\]
If \( w_1 = w_n \) then (3) gives

\[
w_1 = w_2 = \cdots = w_n = \frac{1}{n} \Rightarrow \text{disp}(W) = \ln n,
\]

which is the optimal solution to (1) for \( \alpha = 0.5 \) (actually, this is the global optimal value for the dispersion of all OWA operators of dimension \( n \)). Suppose now that \( w_1 \neq w_n \). Let us introduce the notations

\[
u_1 = w_1^{-\frac{1}{n-1}}, \quad \nu_n = w_n^{-\frac{1}{n-1}}.
\]

Then we may rewrite (3) as \( w_j = \nu_1^{n-j} \nu_n^{-j} \), for \( 1 \leq j \leq n \).

From the first condition, orness\((W) = \alpha \), we find

\[
\sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha \iff \sum_{i=1}^{n} (n-i) \nu_1^{n-i} \nu_n^{-i} = (n-1)\alpha,
\]

and from

\[
\sum_{i=1}^{n} \frac{(n-i)\nu_1^{n-i} \nu_n^{-i}}{n-1} = \frac{1}{\nu_1 - \nu_n} \left[ (n-1)\nu_1^n - \sum_{i=1}^{n-1} \nu_1^{n-i} \right] = \frac{1}{\nu_1 - \nu_n} \left[ (n-1)\nu_1^n - \nu_1 \nu_n \nu_n^{-1} \right] = \frac{1}{(\nu_1 - \nu_n)^2} \left[ (n-1)\nu_1^{n+1} - n\nu_1^n + \nu_1 \nu_n^n \right],
\]

we get

\[
(n-1)\nu_1^{n+1} - n\nu_1^n \nu_n + \nu_1 \nu_n^n = (n-1)\alpha (\nu_1 - \nu_n)^2
\]

\[
\nu_1^n - \nu_n = (n-1)\alpha (\nu_1 - \nu_n)
\]

\[
u_n = \frac{1}{(n-1)\alpha} \left[ ((n-1)\alpha + 1)\nu_1 - \nu_1^n \right]
\]

\[
u_n = \frac{(n-1)\alpha + 1 - nw_1}{(n-1)\alpha}.
\]

(4)
From the second condition, \( w_1 + \cdots + w_n = 1 \), we get
\[
\sum_{j=1}^{n} u_1^{n-j} w_n^{j-1} = 1 \iff \frac{u_1^n - u_n^n}{u_1 - u_n} = 1
\]
\[
\iff u_1^n - u_n^n = u_1 - u_n \quad (5)
\]
\[
\iff u_1^{n-1} - u_n^{n-1} = 1 - \frac{u_n}{u_1} \quad (6)
\]
Comparing equations (4) and (6) we find
\[
w_1 - \frac{(n-1)\alpha + 1 - nw_1}{(n-1)\alpha} \times w_n = \frac{nw_1 - 1}{(n-1)\alpha}
\]
and, therefore,
\[
w_n = \frac{((n-1)\alpha - n)w_1 + 1}{(n-1)\alpha + 1 - nw_1}. \quad (7)
\]
Let us rewrite equation (5) as
\[
u_1^n - u_n^n = u_1 - u_n
\]
\[
u_1(w_1 - 1) = u_n(w_n - 1)
\]
\[
w_1(w_1 - 1)^{n-1} = w_n(w_n - 1)^{n-1}
\]
\[
w_1(w_1 - 1)^{n-1} = \left(\frac{(n-1)\alpha - n}{(n-1)\alpha + 1 - nw_1}\right)^{n-1} \times \left[\frac{(n-1)\alpha(w_1 - 1)}{(n-1)\alpha + 1 - nw_1}\right]^{n-1}
\]
\[
w_1[(n-1)\alpha + 1 - nw_1]^{n} = \left(\frac{(n-1)\alpha}{(n-1)\alpha + 1 - nw_1}\right)^{n-1}[(n-1)\alpha - n]w_1 + 1]. \quad (8)
\]
So the optimal value of \( w_1 \) should satisfy equation (8). Once \( w_1 \) is computed then \( w_n \) can be determined from equation (7) and the other weights are obtained from equation (3).

**Remark 2.1** If \( n = 3 \) then from (3) we get
\[
w_2 = \sqrt{w_1 w_3}
\]
independently of the value of \( \alpha \), which means that the optimal value of \( w_2 \) is always the geometric mean of \( w_1 \) and \( w_3 \).
3 Computing the optimal weights

Let us introduce the notations

\[ f(w_1) = w_1[(n-1)\alpha + 1 - nw_1]^n, \]
\[ g(w_1) = ((n-1)\alpha)^{n-1}[(n-1)\alpha - n]w_1 + 1. \]

Then to find the optimal value for the first weight we have to solve the following equation

\[ f(w_1) = g(w_1), \]

where \( g \) is a line and \( f \) is a polynomial of \( w_1 \) of dimension \( n + 1 \).

Without loss of generality we can assume that \( \alpha < 0.5 \), because if a weighting vector \( W \) is optimal for problem (1) under some given degree of orness, \( \alpha < 0.5 \), then its reverse, denoted by \( W^R \), and defined as

\[ w^R_i = w_{n-i+1} \]

is also optimal for problem (1) under degree of orness \( (1 - \alpha) \). Really, as was shown by Yager [4], we find that

\[ \text{disp}(W^R) = \text{disp}(W) \text{ and orness}(W^R) = 1 - \text{orness}(W). \]

Therefore, for any \( \alpha > 0.5 \), we can solve problem (1) by solving it with level of orness \( (1 - \alpha) \) and then taking the reverse of that solution.

From the equations

\[ f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \text{ and } f'\left(\frac{1}{n}\right) = g'\left(\frac{1}{n}\right) \]

we get that \( g \) is always a tangency line to \( f \) at the point \( w_1 = 1/n \). But if \( w_1 = 1/n \) then \( w_1 = \cdots = w_n = 1/n \) also holds, and that is the optimal solution for \( \alpha = 0.5 \).

Consider the the graph of \( f \). It is clear that \( f(0) = 0 \) and by solving the equation

\[ f'(w_1) = [(n-1)\alpha + 1 - nw_1]^n - n^2w_1[(n-1)\alpha + 1 - nw_1]^{n-1} = 0 \]

we find that its unique solution is

\[ \hat{w}_1 = \frac{(n-1)\alpha + 1}{n(n+1)} < \frac{1}{n}, \]

and its second derivative, \( f''(\hat{w}_1) \) is negative, which means that \( \hat{w}_1 \) is the only maximizing point of \( f \) on the segment \([0, 1/n]\).
We prove now that \( g \) can intersect \( f \) only once in the open interval \((0, 1/n)\). It will guarantee the uniqueness of the optimal solution of problem (1). Really, from the equation

\[
f''(w_1) = -2n^2[(n-1)\alpha+1-nw_1]^{n-1} + n^3(n-1)w_1[(n-1)\alpha+1-nw_1]^{n-2} = 0
\]

we find that its unique solution is

\[
\bar{w}_1 = \frac{2(n-1)\alpha + 1}{n(n+1)} = 2\hat{w}_1 < \frac{1}{n}, \quad \text{since } \alpha < 0.5.
\]

with the meaning that \( f \) is strictly concave on \((0, \bar{w}_1)\), has an inflection point at \( \bar{w}_1 \), and \( f \) is strictly convex on \((\bar{w}_1, 1/n)\). Therefore, the graph of \( g \) should lie below the graph of \( g \) if \( \hat{w}_1 < w_1 < 1/n \) and \( g \) can cross \( f \) only once in the interval \((0, \bar{w}_1)\).

### 4 Illustrations

Let us suppose that \( n = 5 \) and \( \alpha = 0.6 \). Then from the equation

\[
w_1[4 \times 0.6 + 1 - 5w_1]^5 = (4 \times 0.6)^4[1 - (5 - 4 \times 0.6)w_1].
\]

we find

\[
w_1^* = 0.2884
\]

\[
w_5^* = \frac{(4 \times 0.6) - 5)w_1^* + 1}{4 \times 0.6 + 1 - 5w_1^*} = 0.1278
\]

\[
w_2^* = \sqrt{(w_1^*)^3w_5^*} = 0.2353,
\]

\[
w_3^* = \sqrt{(w_1^*)^2(w_5^*)^2} = 0.1920,
\]

\[
w_4^* = \sqrt[4]{(w_1^*)(w_5^*)^3} = 0.1566.
\]

and, \( \text{disp}(W^*) = 1.5692 \).

Using exponential smoothing [1], Filev and Yager [6] obtained the following weighting vector

\[
W' = (0.41, 0.10, 0.13, 0.16, 0.20),
\]
with \( \text{disp}(W') = 1.48 \) and \( \text{orness}(W') = 0.5904 \).

We first note that the weights computed from the constrained optimization problem have better dispersion than those ones obtained by Filev and Yager in [6], however the (heuristic) technology suggested in [6] needs less computational efforts.

Other interesting property here is that small changes in the required level of orness, \( \alpha \), can cause a big variation in weighting vectors of near optimal dispersity, (for example, compare the weighting vectors \( W^* \) and \( W' \)).

References


There are several methods that find the OWA weights. [3,9-12] Yager [3] proposed finding weights that maximizes their entropy (also known as weights dispersion). Later, Fuller and Majlender [A13] found an analytical solution to Yager’s theory based on Lagrange multipliers. In addition, these two authors proposed minimizing the variance of weights [A10] instead of maximizing their entropy. The goal of these methods is to maximize similarity. Maximizing the entropy or minimizing the variance of the weights \( \{w_1, w_2, \ldots, w_n\} \) makes each weight \( w_i \) more similar in magnitude to the others. (page 582)
This step computes the aggregate values of the test-score examples and typical score examples, for different orders of evaluation items by using the OWA operator. First, according to the Fullér and Majlender’s equation introduced in Section 3, we can obtain a set of OWA weights $W_\alpha \{w_1, w_2, \ldots, w_n\}$, where $0 \leq w_1 \leq 1$, $\sum_{i=1}^{n} w_i = 1$ and $\alpha \in [0, 1]$. Second, to compute the aggregate values, we multiply the values of the evaluation items, which are permuted by all possible orders, by the corresponding OWA weights, and then sum up these multiplication values. (page 920)


http://dx.doi.org/10.1016/j.cie.2009.06.007

Additionally, Fuller and Majlender (2001) used Lagrange multipliers on Yager’s OWA equation to derive a polynomial equation, which determines the optimal weighting vector under maximal entropy (ME-OWA operator). The proposed approach thus determines the optimal weighting vector under maximal entropy, and the OWA operator ascertains the optimal reliability allocation rating after an aggregation process. This method is both a simple and effective approach that can efficiently resolve the shortcomings of the FOO technique and average weighting allocation. (page 1275)

http://dx.doi.org/10.1007/s10844-008-0068-1

http://dx.doi.org/10.1142/S1469026809002679

A13-c120 E. Cables Pérez, M.Teresa Lamata, OWA weights determination by means of linear functions, MATHWARE & SOFT COMPUTING, 16(2009), 107-122. 2009

2.1.2. Fullér and Majlender’s OWA

Fullér and Majlender (2001) transform Yager’s OWA equa-
tion to a polynomial equation by using Lagrange multipliers.
According to their approach, the associated weighting vector can be obtained by (5)-(7).

A13-c110 YAO Shuang; GUO Ya-jun; YI Ping-tao, Multi-variable Induced Ordered Weighted Averaging Operator and Its Application, JOURNAL OF NORTHEASTERN UNIVERSITY (NATURAL SCIENCE), 30(2009), number 2, pp. (in Chinese). 2009


http://www.cci.dmu.ac.uk/preprintPDF/Franciscov4i4p5.pdf

http://dx.doi.org/10.1080/03052150802132914

A13-c104 Konstantinos Anagnostopoulos, Haris Doukas, John Psarras, A linguistic multicriteria analysis system combining fuzzy sets theory,

In this paper we use a special class of OWA operators which have maximum entropy for a given level of orness (O’Hagan, 1988). The weighing vector of an OWA operator with maximum entropy is calculated applying the results of Fullér and Majlender (2001). (page 2044)


However to apply the OWA operator for decision making, a very crucial issue is to determine its weights. O’Hagan [16] suggested a maximum entropy method as the rst approach to determine OWA operator weights in which he formulated the OWA operator weight problem to a constrained nonlinear optimization model with a predened degree of orness. Fullér and Majlender [A13] transformed the maximum entropy method into a polynomial equation that can be solved analytically.


A13-c99 Xinwang Liu, A general model of parameterized OWA aggregation with given orness level INTERNATIONAL JOURNAL OF APPROXIMATE REASONING, vol. 48, pp. 598-627. 2008 http://dx.doi.org/10.1016/j.ijar.2007.11.003

Filev and Yager [12] further proposed a method to generate MEOWA weight vector by an immediate parameter. Fullér and Majlender [A13] transformed the maximum entropy model into a polynomial equation, which can be solved analytically. (page 599)
Xinwang Liu and Da Qingli, On the properties of regular increasing monotone (RIM) quantifiers with maximum entropy, INTERNATIONAL JOURNAL OF GENERAL SYSTEMS, Volume 37, Issue 2, pp. 167-179. 2008
http://dx.doi.org/10.1080/03081070701192675

Byeong Seok Ahn, Some Quantifier Functions From Weighting Functions With Constant Value of Orness, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS PART B: CYBERNETICS, 38: (2) 540-546. 2008
http://dx.doi.org/10.1109/TSMCB.2007.912743

Xinwang Liu and Shilian Han, Orness and parameterized RIM quantifier aggregation with OW A operators: A summary, INTERNATIONAL JOURNAL OF APPROXIMATE REASONING, Volume 48, Issue 1, Pages 77-97. 2008
http://dx.doi.org/10.1016/j.ijar.2007.05.006

http://dx.doi.org/10.1016/j.ijar.2007.04.001

The resulting weights are called maximum entropy OWA (ME-OWA) weights for a given degree of orness and analytic forms and property for these weights are further investigated by several researchers [25,A13]. (page 167)

http://dx.doi.org/10.1002/int.20257

http://www.mathnet.or.kr/mathnet/thesis_file/DHHong0613F.pdf

In the field of OW A operators, one of the first approaches, suggested by O’Hagan [18], lies in selecting the vector that maximizes the entropy of the OW A weights for a given level of orness. This methodology has also been used by Fullér and Majlender [A13]. (page 4745)

Fullér and Majlender (2001) showed that the maximum entropy model could be transformed into a polynomial equation that can be solved analytically. (page 203)

http://dx.doi.org/10.1080/01969720601187347

A13-c82  Wang YM, Parkan C, A preemptive goal programming method for aggregating OWA operator weights in group decision making INFORMATION SCIENCES, 177 (8): 1867-1877 APR 15 2007  
http://dx.doi.org/10.1016/j.ins.2006.07.023

Fullér and Majlender [A13] showed that the maximum entropy model could be converted into a polynomial equation that can be solved analytically. (page 1867)

http://dx.doi.org/10.1002/int.20201

Fullér and Majlender [A13] transformed the maximum entropy model into a polynomial equation that can be solved analytically. (page 209)

http://dx.doi.org/10.1016/j.ins.2006.03.001

Fullér and Majlender [A13] used the method of Lagrange multipliers to solve O’Hagan’s procedure analytically. (page 251)

A13-c79  Sadiq, R., Tesfamariam, S. Probability density functions based weights for ordered weighted averaging (OWA) operators: An example
http://dx.doi.org/10.1016/j.ejor.2006.09.041

Yager and Filev (1999) suggested an algorithm to obtain the OWA weights from a collection of samples with the relevant aggregated data. Fullér and Majlender (2001) used the method of Lagrange multipliers to solve O’Hagan’s procedure analytically. (page 1356)

http://dx.doi.org/10.1016/j.ijar.2006.06.004

Recently, Fullér [A13] transformed the maximum entropy model into a polynomial equation, which can be solved in an analytical way. (page 69)

http://dx.doi.org/10.1080/01969720600939797

A13-c76 Xu ZS, A note on linguistic hybrid arithmetic averaging operator in multiple attribute group decision making with linguistic information GROUP DECISION AND NEGOTIATION, 15 (6): 593-604 NOV 2006
http://dx.doi.org/10.1007/s10726-005-9008-4

Fullér and Majlender (2001) used the method of Lagrange multipliers to solve O’Hagan’s procedure analytically. (page 595)

A13-c75 Liu XW, Lou HW Parameterized additive neat OWA operators with different orness levels INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, 21(10): 1045-1072 OCT 2006
http://dx.doi.org/10.1002/int.20176

The maximum entropy OWA operator was first suggested by O’Hagan [40] and later was discussed by Filev and Yager [21] and Fullér and Majlender [A13]. (page 1055)

The consistent condition of geometric (maximum entropy) OWA operator was proved, some properties associated with the orness level are discussed, which extended the results of O’Hagan [14], Filev and Yager [6,7], Fullér and Majlender [A13].

http://dx.doi.org/ 10.1142/S0218488506004102

Ahn BS, On the properties of OWA operator weights functions with constant level of orness, IEEE TRANSACTIONS ON FUZZY SYSTEMS, 14(4): 511-515 AUG 2006
http://dx.doi.org/10.1109/TFUZZ.2006.876741

Xu ZH, Induced uncertain linguistic OWA operators applied to group decision making, INFORMATION FUSION, 7(2): 231-238 JUN 2006
http://dx.doi.org/10.1016/j.inffus.2004.06.005

http://dx.doi.org/10.1142/S021848850600400X

http://dx.doi.org/10.1007/s00500-005-0030-x

Fullér and Majlender [A13] use the method of Lagrange multipliers to transfer Eq. (12) to a polynomial equation, which can determine the optimal weighting vector. By their method, the associated weighting vector is easily obtained by Eqs. (13)-(18). (page 1033)

Xinwang Liu, On the maximum entropy parameterized interval approximation of fuzzy numbers, FUZZY SETS AND SYSTEMS, 157, pp. 869-878. 2006
http://dx.doi.org/10.1016/j.fss.2005.09.010
To resolve this problem, this study proposes a dynamic OWA aggregation model based on the faster OWA operator, which has been introduced by Fullér and Majlender \[A13\] and can work like a magnifying lens and adjust its focus based on the sparsest information to change the dynamic attribute weights to revise the weight of each attribute based on aggregation situation, and then to provide suggestions to decision maker (DM). (page 544)

Comparing the researches on the weights obtaining methods in OWA operator, such as the quantifier guided aggregation \[2, 37\], exponential smoothing \[14\], learning \[25\], especially the maximum entropy method \[16, 28, 38, A13\], the WOWA aggregation methods are relatively rare \[30, 40\]. (pages 118-119)

Fullér and Majlender \[A13\] used the method of Lagrange multipliers to solve problem 12 analytically and got the following:

1. If $n = 2$ then $w_1 = \alpha$ and $w_2 = 1 - \alpha$.
2. If $\alpha = 0$ or $\alpha = 1$ then the associated weighting vectors are uniquely defined as $w = (0, 0, \ldots, 1)^T$ and $w = (1, 0, \ldots, 0)^T$ respectively, with value of dispersion zero.
3. If \( n \geq 3 \) and \( 0 < \alpha < 1 \) then

\[
 w_j = \frac{n-j}{n} w_{n-j}^{j-1} \tag{15}
\]

\[
 w_n = \frac{(n-1)\alpha - n}{(n-1)\alpha + 1 - nw_1} \tag{16}
\]

\[
 w_1[(n-1)\alpha + 1 - nw_1]^n = ((n-1)\alpha - n)w_1 + 1 \tag{17}
\]

Solving Equations 15-17, the optimal OWA weights can be determined. (page 847)

A13-c63 Lan H, Ding Y, Hong J, Decision support system for rapid prototyping process selection through integration of fuzzy synthetic evaluation and an expert system INTERNATIONAL JOURNAL OF PRODUCTION RESEARCH, 43 (1): 169-194 JAN 1 2005

http://dx.doi.org/10.1080/00207540410001733922


http://dx.doi.org/10.1093/pan/mpi002


http://dx.doi.org/10.1016/j.ins.2004.09.003

Fullé and Majlender [A13] showed that the maximum entropy model could be transformed into a polynomial equation that can be solved analytically. (page 21)


A13-c56 Liu XW, On the methods of decision making under uncertainty with probability information, INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, 19(12): 1217-1238 DEC 2004

http://dx.doi.org/10.1002/int.20045
The maximum entropy OWA operator was first suggested by O’Hagan [11] and later was discussed by Filev and Yager [10] and Fullér and Majlender [A13]. (page 1225)

**A13-c55** Xinwang Liu and Lianghua Chen, On the properties of parametric geometric OWA operator, INTERNATIONAL JOURNAL OF APPROXIMATE REASONING, 35 pp. 163-178. 2004
http://dx.doi.org/10.1016/j.ijar.2003.09.001

Recently, Fullér and Majlender [A13] proposed another method to generate MEOWA weights, the method get the weights by solving a polynomial equation. (page 164)

http://dx.doi.org/10.1002/int.10172

**A13-c53** Beliakov G., How to build aggregation operators from data INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, 18(8): 903-923 AUG 2003
http://dx.doi.org/10.1002/int.10120

**A13-c52** Xu, Z., Da, Q. Approaches to obtaining the weights of the ordered weighted aggregation operators Dongnan Daxue Xuebao (Ziran Kexue Ban)/Journal of Southeast University (Natural Science Edition), 33 (1), pp. 94-96. 2003

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http://doi.ieeecomputersociety.org/10.1109/ASONAM.2009.72

http://dx.doi.org/10.1109/CIFER.2009.4937506


http://dx.doi.org/10.1117/12.818669
We therefore formulate the multi-modal fusion as an information aggregation task in the framework of group decision making (GMD) problem. Specifically, we employ the Ordered Weighted Average (OWA) operator to aggregate the group of decisions by uni-modal detectors, as it has been reported to be an effective solution for GMD problem [A13]. (page 209)

In Dispersion Maxspace represented by Eqn (8), the weights of different Orness are given with maximal dispersion which means most individual criteria are being used in the aggregation that gives more robustness [A13]. (page 212)
http://dx.doi.org/10.1007/978-3-540-73723-0_8

http://dx.doi.org/10.1007/978-3-540-73723-0_15

http://doi.ieeecomputersociety.org/10.1109/ICTAI.2007.123

http://dx.doi.org/10.1109/ICMLC.2007.4370360

Fullér and Majlender [A13] transform Yager's OWA equation to a polynomial equation by using Lagrange multipliers. According to their approach, the associated weighting vector can be obtained by (2) - (4). (page 1384)

A13-c15 Wang, Jia-Wen; Cheng, Ching-Hsue, Information Fusion Technique for Weighted Time Series Model, International Conference on
Fullér and Majlender use the method of Lagrange multipliers to transfer equation (7) to a polynomial equation, which can determine the optimal weighting vector. By their method, the associated weighting vector is easily obtained by (8)-(9) [A13].

(page 1861)

http://dx.doi.org/10.1109/ICPCA.2007.4365438

http://dx.doi.org/10.1109/ICMLC.2007.4370452

A13-c12 Ching-Hue Cheng, Jing-Wei Liu, OWA Rough Set to Forecast the Industrial Growth Rate, International Conference on Convergence Information Technology, 21-23 Nov. 2007, pp. 1862-1867. 2007
http://doi.ieeecomputersociety.org/10.1109/ICITC.2007.233

http://dx.doi.org/10.1007/978-3-540-77296-5_29

http://dx.doi.org/10.1007/978-3-540-73721-6_7
Filev and Yager [11] simplified this optimization problem using the Lagrange multipliers method. Then the problem boils down to finding the root of a polynomial of degree $m - 1$. Fullér and Majlender [A13], assuming the same approach, proposed a simpler formulae for the weight vector $W$. (page 101)
Fullér and Majlender [A13] used the method of Lagrange multipliers to transfer Yager’s OWA equation to a polynomial equation, which can determine the optimal weighting vector. By their method, the associated weighting vector is easily obtained by (5)-(7).

\[
\ln w_j = \frac{j - 1}{n - 1} \ln w_n + \frac{n - j}{n - 1} \ln w_1 \Rightarrow w_j = \sqrt[n-1]{w_1^{n-j}w_j^{j-1}} \tag{5}
\]

and

\[
w_n = \frac{((n - 1)\alpha - n)w_1 + 1}{(n - 1)\alpha + 1 - nw_1} \tag{6}
\]

then

\[
w_1[(n - 1)\alpha + 1 - nw_1]^n = ((n - 1)\alpha n - 1)[(n - 1)\alpha - n]w_1 + 1 \tag{7}
\]


Entropy has been generally adopted as a measure of weight dispersion of the OWA operators. O’Hagan [2], in his ground braking work, suggests to select the vector that maximizes the entropy of OWA weights (ME-OWA). Analytical solutions to this problem have been proposed by Filev and Yager [3], and Fullér and Majlender [A13]. (page 82)


http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1259989