Control Structure Selection for Integral Control with Integrity

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Outline

- Control structure selection
  - introductory example
  - importance of correct variable pairing
- The Relative Gain Array
- Shortcomings of the RGA
  - as interaction measure
  - as robustness measure
  - in variable pairing
- Examples of problems in variable pairing by RGA
  - necessity to consider partial control
  - inadequacy of conventional pairing rule
- Integral control with integrity
- The Partial Relative Gain
- Application of PRG to Petlyuk distillation column
  - decentralized control
  - block-decentralized control
- Conclusions
We want to control the total flow rate $F$ and the temperature $T$ by manipulating the flowrates $F_1$ and $F_2$ using a decentralized control structure (i.e., two SISO control loops).

The basic question is: How should we pair the variables in the control loops?

Should we control $F$ by $F_1$ and $T$ by $F_2$, or vice versa?
Control Structure Selection
Importance of Correct Variable Pairing

\[ v_H \]

\[ m_H T_H \]

\[ m_D T_D \]

\[ m_C T_C \]

\[ \{ M T \} \]

\[ h \]

\[ m T \]

\[ h_o \]
SISO PI control of mixing tank with correct variable pairing ($\lambda = 0.68$).
SISO PI control of mixing tank with incorrect variable pairing ($\lambda = 0.38$).
The Relative Gain Array
Properties of RGA

Relative gain analysis is a widely used technique in control structure selection. It is based on a “Relative Gain Array” (RGA), which is a matrix of interaction measures for all possible single-input single-output (SISO) pairings between a set of variables. The RGA

- indicates the preferable variable pairings in a decentralized (multiloop SISO) control system based on interaction considerations;

- provides information about fundamental properties such as integral controllability, integrity, and robustness with respect to modelling errors and input uncertainty;

- is not a true measure of closed-loop interactions, which means that the RGA may fail for systems larger than $2 \times 2$. 

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Definition of RGA

Static (non-linear) model

\[
y_1 = f_1(u_1, u_2, \ldots, u_n) \\
y_2 = f_2(u_1, u_2, \ldots, u_n) \\
\vdots \\
y_n = f_n(u_1, u_2, \ldots, u_n)
\]

\[
\lambda_{ij} = \left( \frac{\partial y_i}{\partial u_j} \bigg|_{u_k \neq j} \right) / \left( \frac{\partial y_i}{\partial u_j} \bigg|_{y_k \neq i} \right)
\]

\[
\Lambda = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn}
\end{bmatrix}
\]
Transfer function model

\[ y_1(s) = G_{11}(s)u_1(s) + G_{12}(s)u_2(s) + \ldots + G_{1n}(s)u_n(s) \]

\[ y_2(s) = G_{21}(s)u_1(s) + G_{22}(s)u_2(s) + \ldots + G_{2n}(s)u_n(s) \]

\[ \vdots \]

\[ y_n(s) = G_{n1}(s)u_1(s) + G_{n2}(s)u_2(s) + \ldots + G_{nn}(s)u_n(s) \]

\[ G(s) = \begin{bmatrix}
G_{11}(s) & G_{12}(s) & \ldots & G_{1n}(s) \\
G_{21}(s) & G_{22}(s) & \ldots & G_{2n}(s) \\
\vdots & \vdots & \ddots & \vdots \\
G_{n1}(s) & G_{n2}(s) & \ldots & G_{nn}(s)
\end{bmatrix} \]

\[ \Lambda(s) = G(s) \circ (G(s)^{-1})^T \]

\[ \circ = \text{Hadamard product} = \text{Matlab } \ast \text{ product} \]

\[ (C = A \circ B \iff C_{ij} = A_{ij}B_{ij}) \]
Interpretations of the RGA

The open-loop gain $g_{ij}$ will change by the factor $\lambda_{ij}^{-1}$, where $\lambda_{ij}$ is the relative gain for pairing output “$i$” with input “$j$”, when other control loops are closed. This implies:

- Variable pairings with positive $\lambda_{ij}$ as close to unity as possible should be preferred; $\lambda_{ij} = 1$ indicates a “perfect” variable pairing.

- Negative relative gains should be avoided; $\lambda < 0$ results in a closed-loop system which is only conditionally stable, at best.

- Relative gains much larger than unity should be avoided; a system with $\lambda_{ij} \gg 1$ may be practically uncontrollable.

- If $g_{ij}$ and $\lambda_{ij} = 0$, the relative gain does not indicate whether the variable pairing is feasible; control depends entirely on other control loops.
Shortcomings of the RGA

Limitations of RGA as Interaction Measure

- $\lambda_{ij}$ relates the open-loop gain between $y_i$ and $u_j$ when other outputs are uncontrolled to the same open-loop gain when other outputs are controlled.

  $\Rightarrow$ $\lambda_{ij}$ is an open-loop measure for the $y_i$–$u_j$ pairing.

- $\lambda_{ij}$ does not contain explicit information on how other inputs $u_k$, $k \neq j$, affect $y_i$ when $y_i$ is paired with $u_j$, or how $u_j$ affects other outputs $y_k$, $k \neq i$, when the loop $y_i$–$u_j$ is closed.

  $\Rightarrow$ There may be considerable interaction between control loops even when $\lambda_{ij} = 1$. 
Interaction in Triangular System

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} = \frac{1}{10s + 1} \begin{bmatrix}
  1 & 10 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

\[\Lambda = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}\]

Setpoint change of (a) \(y_1\), (b) \(y_2\). \(y_1 = --\), \(y_2 = --\).
RGA as Robustness Measure

It has been proved that the RGA is a rigorous measure of the sensitivity of the system to modelling errors.

- The system becomes singular if a transfer function $G_{ij}$ is changed by the amount $-G_{ij}\lambda_{ij}^{-1}$.

- If there are perturbations in several transfer functions, the system may become singular even for smaller changes.

- A system with large entries in the RGA ($|\lambda_{ij}| \gg 1$) is very sensitive to modelling errors.

- $\lambda_{ij}^{-1}$ is a measure of the relative error in $G_{ij}$ that will make the system singular; if $G_{ij} \approx 0$, a small absolute perturbation of $G_{ij}$ may completely change the system properties even if $|\lambda_{ij}|$ is small.
Sensitivity of (almost) Triangular System

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} = \frac{1}{10s + 1} \begin{bmatrix}
  1 & 10 \\
  0.09 & 1
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

\[\Lambda = \begin{bmatrix}
  10 & -9 \\
  -9 & 10
\end{bmatrix}\]

Setpoint change of (a) \(y_1\), (b) \(y_2\). \(y_1 = \_\_\_\, y_2 = \_\_\_.\)
Limitations of RGA in Variable Pairing

• The steady-state RGA may suggest a different set of variable pairings than the RGA evaluated at higher frequencies.
  ⇒ It has to be considered what frequency range is most important for feedback control.

• For $3 \times 3$ or larger systems, there is not always a clearcut choice of variable pairings even if only a single frequency is considered.
  o It may be necessary to select variable pairings corresponding to zero or negative relative gains.
  o The best control performance is not necessarily obtained by the set of variable pairings with relative gains closest to unity.
  ⇒ The feasibility of the variable pairings have to be decided by other means.
• Even if all variable pairings correspond to positive relative gains, decentralized integral controllability cannot necessarily be guaranteed.

⇒ The RGA is usually used together with other measures, such as the Niederlinski index or the Morari index of integral controllability.

○ Another solution: RGA for partially controlled plant.
Problems in Variable Pairing by RGA

Example 1: Necessity to consider partially controlled system

Two-product distillation column with total condenser (Häggblom and Waller, 1991):

\[
\begin{bmatrix}
  y(s) \\
  z(s)
\end{bmatrix} =
\begin{bmatrix}
  G_{yu}(s) & 0 \\
  G_{zu}(s) & Is^{-1}
\end{bmatrix}
\begin{bmatrix}
  u(s) \\
  v(s)
\end{bmatrix}
\]

\[
y = [x_D \ x_B]^T, \quad u = [L \ V]^T
\]

\[
z = [h_D \ h_B]^T, \quad v = [D \ B]^T
\]

RGA:

\[
\Lambda(s) = \begin{bmatrix}
  G_{yu}(s) & 0 \\
  G_{zu}(s) & Is^{-1}
\end{bmatrix} \circ \begin{bmatrix}
  G_{yu}(s) & 0 \\
  G_{zu}(s) & Is^{-1}
\end{bmatrix}^{-T}
= \begin{bmatrix}
  \Lambda_{yu}(s) & 0 \\
  0 & I
\end{bmatrix}
\]

where

\[
\Lambda_{yu}(s) = G_{yu}(s) \circ G_{yu}(s)^{-T}
\]

\(\Lambda(s)\) implies that \(y\) should always be controlled by \(u\) and \(z\) by \(v\) (i.e., the LV-structure).
Two-Product Distillation Column

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RGs for Distillation Control Structures


Note: The “best” control structures ($RB$, $LB$, $DB$ at higher frequencies) have variable pairings on zero relative gains of the full open-loop model.
Variable Pairing on $\lambda = 0$

- The feasibility of a variable pairing corresponding to a relative gain equal to zero is entirely dependent on other control loops in the closed-loop system.

  $\Rightarrow$ At least some of the other control loops have to be closed when the feasibility of such variable pairings are considered.

- In distillation, inventory control is necessary to make continuous operation possible, but apart from that, the inventory control is of minor importance compared to product quality control.

  $\Rightarrow$ It is natural to consider product quality control under the assumption that inventory control loops are closed.

- Closing control loops is likely to be a useful procedure also in other situations, where conventional application of RGA does not solve the variable pairing problem.
Example 2: Inadequacy of conventional variable pairing rule

Example by Hovd and Skogestad (1992):

\[ G(s) = \frac{(1 - s)}{(1 + 5s)^2} \begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ \Lambda(G(s)) = \begin{bmatrix} 1 & 5 & -5 \\ -5 & 1 & 5 \\ 5 & -5 & 1 \end{bmatrix} \]

Variable pairing on

- \( \lambda_{ij} = 1 \) (configuration \( C_{123} \)) results in a closed-loop time constant \( \tau_{CL} \approx 1160 \),

- \( \lambda_{ij} = 5 \) (configuration \( C_{231} \)) results in a closed-loop time constant \( \tau_{CL} \approx 220 \).
Partial Control

Example 2 (cont’d)

• Variable pairing on $\lambda_{ij} = 1$ (config. $C_{123}$). Closing control loop $y_3-u_3$ results in the transfer matrix

$$\tilde{G}_{120}(s) = \frac{(1-s)}{(1+5s)^2} \begin{bmatrix} 26.96 & 21.77 \\ 32.15 & 26.96 \end{bmatrix}$$

for the remaining subsystem. The RGA for the partially controlled system is

$$\Lambda(\tilde{G}_{120}(s)) = \begin{bmatrix} 26.98 & -25.98 \\ -25.98 & 26.98 \end{bmatrix}$$

⇒ Control performance will be very sluggish.

• Variable pairing on $\lambda_{ij} = 5$ (config. $C_{231}$). Closing control loop $y_3-u_1$ results in a partially controlled system with the RGA

$$\Lambda(\tilde{G}_{230}(s)) = \begin{bmatrix} 6.19 & -5.19 \\ -5.19 & 6.19 \end{bmatrix}$$

⇒ No particular control problems.
RGA for Decomposed System

Linear $n \times n$ system:

$$y(s) = G(s)u(s)$$

The Relative Gain Array

$$\Lambda(G) = G \circ G^{-T}$$

Let $G$ and $\Lambda$ be partitioned as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

where $G_{22}$ is assumed to be nonsingular. Then

$$\Lambda_{11}(G) = G_{11} \circ \bar{G}_{11}^{-T}$$

where

$$\bar{G}_{11} = G_{11} - G_{12}G_{22}^{-1}G_{21}$$

is the Schur complement of $G_{22}$.

Note: $\bar{G}_{11}$ is also the effective gain matrix of subsystem $G_{11}(s)$ when the rest of the system (i.e., $G_{22}(s)$) is closed under integral feedback control.
The Relative Gain Array (cont’d)

Let $G_m$ denote a square submatrix of $G$ and $\Lambda_m(G)$ the corresponding submatrix of $\Lambda(G)$. Then

$$\Lambda_m(G) = G_m \circ \bar{G}_m^{-T}$$

where $\bar{G}_m$ is the effective gain matrix of subsystem $G_m(s)$ when the rest of the system is closed under integral feedback control.

The RGA for subsystem $G_m(s)$ with the rest of the system open is

$$\Lambda(G_m) = G_m \circ G_m^{-T}$$

The Block Relative Gain

The BRG for MIMO control of subsystem $G_m(s)$ is defined

$$\Lambda^B_m(G) = G_m \bar{G}_m^{-1}$$
Integral Controllability and Integrity


A system \( G(s) \) is integral controllable with integrity (ICI) if there exists a decentralized controller with integral action such that the closed-loop system is unconditionally stable and remains stable when individual controllers are arbitrarily brought in and out of service.

The closed-loop system has this property if it remains stable when the gains of all individual controllers are simultaneously detuned by a factor \( \epsilon \) in the range \( 0 < \epsilon \leq 1 \) as well as when the gains of any combination of controllers are set to 0.

**Theorem ICI**

For variable pairing along the diagonal, \( G(s) \) is ICI only if \( N(G) > 0 \) and \( N(G_{mk}) > 0 \) for all principal submatrices \( G_{mk} \) of size \( k \times k, \ k = 2, \ldots, n - 1 \).

An equivalent condition is \( \lambda_{ii}(G) > 0, \ i = 1, \ldots, n, \) and \( \lambda_{ii}(G_{mk}) > 0, \ i = 1, \ldots, k \).

\[ (\text{Niederlinski index} \ N(G) = \det(G)/\prod_i g_{ii}) \]
Decentralized Integral Controllability


A system \( G(s) \) is decentralized integral controllable (DIC) if there exists a decentralized controller with integral action in each loop, such that the closed-loop system remains stable when the gains of any combination of individual controllers are detuned by individual factors \( \epsilon_i, 0 \leq \epsilon \leq 1 \).

Remarks

- Usually DIC is a desired property since it allows the individual controllers to be arbitrarily detuned.

- The known tests for DIC tend to be rather complicated, however. Furthermore, they are, in general, only necessary but not sufficient, or sufficient but conservative.

- Since DIC implies ICI according to the definitions, Theorem ICI gives necessary conditions for DIC.
The Partial Relative Gain

Let $\tilde{G}_m(s)$ denote the transfer matrix of subsystem $G_m(s)$ when the rest of the system $G(s)$ is under integral feedback control. The partial relative gain (PRG) for subsystem $G_m(s)$ is then

$$\Lambda_m^P(G) = \Lambda(\tilde{G}_m) = \tilde{G}_m \circ \tilde{G}_m^{-T}$$

The PRG provides information that other measures do not:

$$\Lambda_m^P(G) \neq \Lambda_m(G) = G_m \circ G_m^{-T}$$
$$\neq \Lambda(G_m) = G_m \circ G_m^{-T}$$
$$\neq \Lambda_m^B(G) = G_m \tilde{G}_m^{-1}$$

Theorem PRG

For variable pairing along the diagonal, $G(s)$ is ICI only if $\lambda_{ii}(G') > 0$, $i = 1, \ldots, n$, and $\lambda_{ii}(\tilde{G}_{m_k}) > 0$, $i = 1, \ldots, k$, for all principal subsystems $G_{m_k}(s)$ of size $k \times k$, $k = 2, \ldots, n - 1$.

If, in addition to $\lambda_{ii}(G) > 0$, the Niederlinski index $N(G') > 0$, the condition for $k = 2$ is redundant.
Proof of Theorem PRG (Partial)

Consider a system $G(s)$ and a principal subsystem $G_m(s)$ containing the $i$th input and output of $G(s)$. Denote by $G^m_{ii}(s)$ the subsystem obtained by excluding the variables of $G_m(s)$, except the $i$th ones.

\[
 G = \begin{array}{c}
 G_m \\
 g_{ii} \\
 G^m_{ii}
\end{array}
\]

**RGA:** \[ \Lambda_m(G) = G_m \circ \bar{G}_m^{-T} \]

**PRG:** \[ \Lambda(\bar{G}_m) = \bar{G}_m \circ \bar{G}_m^{-T} \]

\[ \Rightarrow G_m \circ \Lambda(\bar{G}_m) = \Lambda_m(G) \circ \bar{G}_m \]

\[ \Rightarrow g_{ii} \lambda_{ii}(\bar{G}_m) = \lambda_{ii}(G) (\bar{G}_m)_{ii} \]

$(\bar{G}_m)_{ii}$ is the effective gain of $g_{ii}$ when the system excluding $G_m$ is closed. \[ \Rightarrow \]

\[ \lambda_{ii}(G^m_{ii}) = g_{ii}/(\bar{G}_m)_{ii} \]

Combination of the last two equations gives

\[ \lambda_{ii}(\bar{G}_m) = \lambda_{ii}(G)/\lambda_{ii}(G^m_{ii}) \]

\[ \Rightarrow \text{RGA conditions of Theorem ICI and Theorem PRG are equivalent.} \]
Case Study: Petlyuk Distillation Column
(Wolff and Skogestad, 1995)

\[ y = \begin{bmatrix} x_{D_1} & x_{B_3} & x_{S_1} & x_{S_2} \end{bmatrix}^T \]
\[ u = \begin{bmatrix} L & V & R_L & S \end{bmatrix}^T \]
Model and RGA:

\[
G = \begin{bmatrix}
153.45 & -179.34 & 0.23 & 0.03 \\
-157.67 & 184.75 & -0.10 & 21.63 \\
24.63 & -28.97 & -0.23 & -0.10 \\
-4.80 & 6.09 & 0.13 & -2.41
\end{bmatrix}
\]

\[
\Lambda(G) = \begin{bmatrix}
24.5230 & -23.6378 & 0.1136 & 0.0012 \\
-48.9968 & 49.0778 & 0.0200 & 0.8990 \\
38.5591 & -38.6327 & 1.0736 & 0.0000 \\
-13.0852 & 14.1927 & -0.2072 & 0.0998
\end{bmatrix}
\]

Potential control configurations according to RGA:

<table>
<thead>
<tr>
<th>Config.</th>
<th>Variable Pairings</th>
<th>Relative Gains</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{1432})</td>
<td>{1–1,2–4,3–3,4–2}</td>
<td>{24.5, 0.90, 1.07, 14.2}</td>
</tr>
<tr>
<td>(C_{3412})</td>
<td>{1–3,2–4,3–1,4–2}</td>
<td>{0.11, 0.90, 38.6, 14.2}</td>
</tr>
<tr>
<td>(C_{1234})</td>
<td>{1–1,2–2,3–3,4–4}</td>
<td>{24.5, 49.1, 1.07, 0.10}</td>
</tr>
<tr>
<td>(C_{3214})</td>
<td>{1–3,2–2,3–1,4–4}</td>
<td>{0.11, 49.1, 38.6, 0.10}</td>
</tr>
<tr>
<td>(C_{1342})</td>
<td>{1–1,2–3,3–4,4–2}</td>
<td>{24.5, 0.02, 0.00, 14.2}</td>
</tr>
<tr>
<td>(C_{4312})</td>
<td>{1–4,2–3,3–1,4–2}</td>
<td>{0.00, 0.02, 38.6, 14.2}</td>
</tr>
</tbody>
</table>
Application of PRG

Closing loop 1–1 gives:

\[
\tilde{G}_{0234} = \begin{bmatrix}
0 & 0.4780 & 0.1363 & 21.6608 \\
0 & -0.1844 & -0.2669 & -0.1048 \\
0 & 0.4081 & 0.1372 & -2.4091 \\
\end{bmatrix}
\]

\[
\Lambda(\tilde{G}_{0234}) = \begin{bmatrix}
0 & 0.1270 & -0.0272 & 0.9003 \\
0 & -0.2460 & 1.2460 & 0.0000 \\
0 & 1.1190 & -0.2187 & 0.0997 \\
\end{bmatrix}
\]

Theorem PRG: \( C_{1342} \) is not ICI. \( C_{1432} \) appears very good.

Closing loop 3–1 gives:

\[
\Lambda(\tilde{G}_{2304}) = \begin{bmatrix}
0.1515 & 0.8215 & 0.0270 \\
-0.1867 & 0.3143 & 0.8724 \\
1.0352 & -0.1358 & 0.1006 \\
\end{bmatrix}
\]

\( \Rightarrow \) \( C_{3214} \) is not ICI, \( C_{3412} \) appears very good.
Closing loop 4–2 gives:

\[ \Lambda(\bar{G}_{1340}) = \begin{bmatrix}
1.9334 & 2.0048 & -2.9382 \\
-3.7459 & 0.8083 & 3.9376 \\
2.8125 & -1.8131 & 0.0006 \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix} \]

\[ \Rightarrow C_{4312} \text{ and } C_{1432} \text{ (!) are not ICI.} \]

Configurations \( C_{1234} \) and \( C_{3412} \) pass all PRG tests for ICI.

Closing loops \{1–1, 3–3\} in \( C_{1234} \) and \{1–3, 3–1\} in \( C_{3412} \) gives:

\[ \Lambda(\bar{G}_{0204}) = \Lambda(\bar{G}_{0402}) = \begin{bmatrix}
0.1020 & 0.8980 \\
0.8980 & 0.1020
\end{bmatrix} \]

Partial relative gains: 0.8980 for \( C_{3412} \), 0.1020 for \( C_{1234} \).

\[ \Rightarrow C_{3412} \text{ should be preferred over } C_{1234}. \]
Control of Petlyuk column, configuration $C_{3412}$. (Decentralized PI control designed for $T_r = 5$ min; first order column dynamics, $T = 20$ min.)
Control of Petlyuk column, configuration $C_{1234}$. (Decentralized PI control designed for $T_r = 5$ min; first order column dynamics, $T = 20$ min.)
Control of Petlyuk column, configuration $C_{1432}$. (Decentralized PI control designed for $T_r = 5$ min; first order column dynamics, $T = 20$ min.)
Block decentralized control

Diagonal BRG elements for $2 \times 2$ subsystems: $\text{RGA}(:, [1 1 1 2 2 3]) + \text{RGA}(:, [2 3 4 3 4 4]) \Rightarrow$

<table>
<thead>
<tr>
<th></th>
<th>$u_1 + u_2$</th>
<th>$u_1 + u_3$</th>
<th>$u_1 + u_4$</th>
<th>$u_2 + u_3$</th>
<th>$u_2 + u_4$</th>
<th>$u_3 + u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.89</td>
<td>24.6</td>
<td>24.5</td>
<td>-23.5</td>
<td>-23.6</td>
<td>0.11</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.08</td>
<td>-49.0</td>
<td>-48.1</td>
<td>49.1</td>
<td>50.0</td>
<td>0.92</td>
</tr>
<tr>
<td>$y_3$</td>
<td>-0.07</td>
<td>39.6</td>
<td>38.6</td>
<td>-37.6</td>
<td>-38.6</td>
<td>1.07</td>
</tr>
<tr>
<td>$y_4$</td>
<td>1.11</td>
<td>-13.3</td>
<td>-13.0</td>
<td>14.0</td>
<td>14.3</td>
<td>-0.11</td>
</tr>
</tbody>
</table>

Implies variable pairings $(y_1, y_4)-(u_1, u_2)$ and/or $(y_2, y_3)-(u_3, u_4)$. Which ones?

PRGs for the remaining system when $(y_1, y_4)-(u_1, u_2)$ and $(y_2, y_3)-(u_3, u_4)$ are closed (one at a time) $\Rightarrow$

$$\Lambda(\bar{G}_{0340}) = \begin{bmatrix} 0.00 & 1.00 \\ 1.00 & 0.00 \end{bmatrix}, \quad \Lambda(\bar{G}_{1002}) = \begin{bmatrix} 28 & -27 \\ -27 & 28 \end{bmatrix}$$

The PRGs imply that configuration

- $C_{0430}$ (pairings $\{y_2-u_4, y_3-u_3, (y_1, y_4)-(u_1, u_2)\}$) is excellent

- $C_{1002}$ (pairings $\{y_1-u_1, y_4-u_2, (y_2, y_3)-(u_3, u_4)\}$) is poor

NERPPI Course 1999: Control Structure Selection... 36
Control of Petlyuk column, configuration $C_{0430}$. (Block-decentralized PI control designed for $T_r = 5$ min; first order column dynamics, $T = 20$ min.)
Control of Petlyuk column, configuration $C_{1002}$. (Block-decentralized PI control designed for $T_r = 5$ min; first order column dynamics, $T = 20$ min.)
Integrated square errors

Integrated square errors of simulated control configurations \( (t = 1000 \text{ min}) \):

<table>
<thead>
<tr>
<th>Config.</th>
<th>( e_y^2 )</th>
<th>( e_u^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{3412} )</td>
<td>360</td>
<td>17025</td>
</tr>
<tr>
<td>( C_{1234} )</td>
<td>16409</td>
<td>4206</td>
</tr>
<tr>
<td>( C_{1432} )</td>
<td>15592</td>
<td>7143</td>
</tr>
<tr>
<td>( C_{0430} )</td>
<td>15</td>
<td>669</td>
</tr>
<tr>
<td>( C_{1002} )</td>
<td>15483</td>
<td>6900</td>
</tr>
</tbody>
</table>

\( e_y^2 \) = sum of ISE of all outputs  
\( e_u^2 \) = sum of ISE of all inputs
Conclusions

- Control structure selection with an emphasis on decentralized integral control and integrity was considered.
- Some fundamental shortcomings of the (open-loop) RGA – a popular tool for control structure selection – were mentioned.
- In particular, variable paring based on the RGA is unreliable for systems larger than $2 \times 2$.
- It was shown that many variable pairing problems could be solved by considering the RGA for a partially controlled system.
- The Partial Relative Gain (PRG) provides necessary conditions for integral controllability with integrity.
- The PRG can solve the variable pairing problem when conventional use of the RGA fails or is ambiguous.
- The PRG is also useful when considering block-decentralized control.
$F = F_1 + F_2$

$T = (T_1 F_1 + T_2 F_2)/F$

$G_{FF_1} = 1, \quad G_{FF_2} = 1$

$G_{TF_1} = (T_1 - T_2)/F, \quad G_{TF_2} = (T_2 - T_1)/F$

$\lambda_{FF_1} = (T - T_2)/(T_1 - T_2) = F_1/F$
RGA for $2 \times 2$ System

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\lambda_{11}(s) = \left( 1 - \frac{G_{12}(s)G_{21}(s)}{G_{11}(s)G_{22}(s)} \right)^{-1}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{bmatrix}$$
Applications of the PRG

Example 3: Elimination of infeasible configurations

Example by Campo and Morari (1994):

\[
G = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0.5 & 2 \\
0.25 & 0.5 & 0.25 & 4 \\
4 & 2 & 0.25 & 4 \\
\end{bmatrix}
\]

\[
\Lambda(G) = \begin{bmatrix}
0.8000 & -2.0000 & 2.1333 & 0.0667 \\
-2.0000 & 5.0000 & -1.3333 & -0.6667 \\
0.0667 & -0.6667 & 0.1778 & 1.4222 \\
2.1333 & -1.3333 & 0.0222 & 0.1778 \\
\end{bmatrix}
\]

Configurations that may be ICI according to the RGA: \(C_{1234}, C_{1243}, C_{3214}, C_{3241}, C_{4213}\) and \(C_{4231}\).

Configurations \(C_{1243}, C_{3214}\) and \(C_{4231}\) can be eliminated because they have negative Niederlinski indices.
Example 3 (cont’d)

Closing control loop $y_2-u_2$:

$$
\bar{G}_{1034} = \begin{bmatrix}
0 & : & 0.5 & -1 \\
\vdots & & \square & \cdots & \cdots \\
-0.25 & : & 0 & 3 \\
2 & : & -0.75 & 0
\end{bmatrix}
$$

$$
\Lambda(\bar{G}_{1034}) = \begin{bmatrix}
0 & : & 1.0667 & -0.0667 \\
\cdots & & \square & \cdots & \cdots & \cdots \\
-0.0667 & : & 0 & 1.0667 \\
1.0667 & : & -0.0667 & 0
\end{bmatrix}
$$

Only configuration $C_{3241}$ may be ICI.

Configuration $C_{3241}$ passes all tests for ICI based on Theorem PRG.
Applications of the PRG

Example 4: Ranking of feasible configurations

Example by Niederlinski (1971), Mijares et al. (1986):

$$G = \begin{bmatrix}
1.0 & -0.1 & 1.0 \\
-0.5 & 0.6 & 0.1 \\
-0.2 & -0.8 & 0.3
\end{bmatrix}$$

$$\Lambda(G) = \begin{bmatrix}
0.3390 & -0.0169 & 0.6780 \\
0.5020 & 0.3911 & 0.1069 \\
0.1591 & 0.6258 & 0.2151
\end{bmatrix}$$

Configurations with variable pairings on positive relative gain values: $C_{123}, C_{132}, C_{312}, C_{321}$.

All configurations satisfy the ICI requirements of Theorem PRG.
Example 4 (cont’d)

Comparison of $C_{123}$ and $C_{132}$ by closing loop $y_1-u_1$:

$$
\bar{G}_{023} = \begin{bmatrix} 0.55 & 0.60 \\ -0.82 & 0.50 \end{bmatrix},
$$

$$
\Lambda(\bar{G}_{023}) = \begin{bmatrix} 0.3585 & 0.6415 \\ 0.6415 & 0.3585 \end{bmatrix}
$$

The PRG implies $C_{132} > C_{123}$.

By closing other loops it can be shown that

$$
C_{312} > \{C_{132}, C_{321}\} > C_{123}
$$
Applications of the PRG

Example 5: Block-decentralized control

Example studied by Alatiqi and Luyben (1986), Grosdidier and Morari (1987):

\[
G = \begin{bmatrix}
4.09 & -6.36 & -0.25 & -0.49 \\
-4.17 & 6.93 & -0.05 & 1.53 \\
-1.73 & 5.11 & 4.61 & -5.48 \\
-11.28 & 14.04 & -0.10 & 4.49 \\
\end{bmatrix}
\]

\[
\Lambda(G) = \begin{bmatrix}
3.1058 & -0.9007 & -0.4749 & -0.7302 \\
-5.0308 & 4.6742 & -0.0395 & 1.3961 \\
-0.0838 & 0.0543 & 1.5492 & -0.5197 \\
3.0088 & -2.8278 & -0.0348 & 0.8538 \\
\end{bmatrix}
\]

Only configuration with all variable pairings on positive relative gain values is \(C_{1234}\).

\(C_{1234}\) satisfies all necessary ICI conditions of Theorem PRG.
Example 5 (cont’d)

Closing loop $y_2-u_2$:

\[
\Lambda(G_{1034}) = \begin{bmatrix}
0.1997 & -0.5620 & 1.3623 \\
\cdots & \cdot & \cdots \\
0.0651 & 1.5616 & -0.6267 \\
0.7352 & 0.0005 & 0.2644
\end{bmatrix}
\]

The PRG suggests that the variable pairings \( \{y_1-u_1, y_4-u_4\} \) are inferior to \( \{y_1-u_4, y_4-u_1\} \).

Closing loop $y_3-u_3$ in addition to $y_2-u_2$:

\[
\Lambda(G_{1004}) = \begin{bmatrix}
0.2647 & \cdot & \cdot & 0.7353 \\
\cdots & \cdot & \cdots \\
\cdots & \cdot & \cdot & \cdots \\
0.7353 & \cdot & \cdot & 0.2647
\end{bmatrix}
\]

This PRG implies that configuration $C_{4231}$ would be better than $C_{1234}$.

However, $C_{4231}$ is not ICI (NI < 0).

The contradiction suggests a block-decentralized control structure with variable pairings \( \{y_2-u_2, y_3-u_3, (y_1, y_4)-(u_1, u_4)\} \).