Modelling, linearization, and simulation of two interacting tanks

Consider the process consisting of two interacting liquid tanks in the figure. The volumetric flow rate into tank 1 is \( F_0 \), the vol. flow rate from tank 1 to tank 2 is \( F_1 \), and the vol. flow rate from tank 2 is \( F_2 \). The height of the liquid level is \( h_1 \) in tank 1 and \( h_2 \) in tank 2. Both tanks have the same cross-sectional area \( A \). The flow rates \( F_1 \) and \( F_2 \) depend on the liquid levels according to

\[ F_1 = \beta \sqrt{h_1 - h_2}, \quad F_2 = \beta \sqrt{h_2}, \]

where \( \beta \) is a constant parameter.

The system can be described by the two coupled first-order nonlinear differential equations

\[
\begin{align*}
\frac{dh_1}{dt} &= \frac{1}{A} F_0 - \frac{\beta}{A} \sqrt{h_1 - h_2}, \\
\frac{dh_2}{dt} &= \frac{\beta}{A} \sqrt{h_1 - h_2} - \frac{\beta}{A} \sqrt{h_2},
\end{align*}
\]

where \( h_1 \geq h_2 \) is assumed.

Linearization of the system equations at a steady-state operating point \( (\bar{h}_1, \bar{h}_2, \bar{F}_0) \) yields

\[
\begin{align*}
\frac{d\Delta h_1}{dt} &= \frac{1}{A} \Delta F_0 - \frac{\beta}{2A \sqrt{\bar{h}_1 - \bar{h}_2}} \Delta h_1 + \frac{\beta}{2A \sqrt{\bar{h}_1 - \bar{h}_2}} \Delta h_2, \\
\frac{d\Delta h_2}{dt} &= \frac{\beta}{2A \sqrt{\bar{h}_1 - \bar{h}_2}} \Delta h_1 - \frac{\beta}{2A \sqrt{\bar{h}_1 - \bar{h}_2}} \left( \frac{1}{\sqrt{\bar{h}_1 - \bar{h}_2}} + \frac{1}{\sqrt{\bar{h}_2}} \right) \Delta h_2,
\end{align*}
\]

where \( \Delta h_1 \equiv h_1 - \bar{h}_1, \Delta h_2 \equiv h_2 - \bar{h}_2 \), and \( \Delta F_0 = F_0 - \bar{F}_0 \) are “Δ-variables”.

a) Show that the linearization of equations (1) yields equations (2). Determine the numerical values of the coefficients in front of the Δ-variables when \( A = 0.15 \text{ m}^2 \), \( \beta = 2.5 \text{ m}^2/\text{h} \), \( \bar{h}_1 = 2 \text{ m} \), \( \bar{h}_2 = 1 \text{ m} \), \( \bar{F}_0 = 2.5 \text{ m}^3/\text{h} \).

b) Simulate the non-linear system (1) and the linearized system (2) for a step change

\[
F_0(t) = \begin{cases} 
2.5 \text{ m}^3/\text{h} & t < 1 \text{ h} \\
3.5 \text{ m}^3/\text{h} & t \geq 1 \text{ h}
\end{cases}
\]

Use the MATLAB software Simulink for the simulation. Use the parameters given in a). Add the steady-state values \( \bar{h}_1 \) and \( \bar{h}_2 \) to the simulated outputs \( \Delta h_1 \) and \( \Delta h_2 \) to get \( h_1 \) and \( h_2 \), respectively. Note also that \( \Delta F_0 \) is an input to (2), not \( F_0 \).

c) Use MATLAB’s plot command to plot how \( h_1 \) and \( h_2 \) change as functions of time for both simulations. You can get help on the plot command by typing “help plot” in MATLAB’s command window.

d) Present your results in a nice report. You can submit it on paper or electronically.