Step and frequency responses using Laplace transform methods

1. a) Consider the analog low-pass filter in the figure. In Example 3.1 in the lecture notes, the model (using other notation)

\[ RC \frac{du_1(t)}{dt} + u_1(t) = u_0(t) \]  \hspace{1cm} (1)

was derived under the assumption that the filter is uncharged on the output side. This means that \( i_1 = 0 \). Assume that \( RC = 1 \) sec, and calculate by using the Laplace transform how the output \( u_1 \) changes with time \( t \) when there is a unit step change in \( u_0 \), i.e.,

\[ u_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \]

b) If two identical filters are connected in series as shown by the figure below, it might seem obvious that the combined filter can be described by (1) when it is uncharged on the output side \( u_2 \). However, this is incorrect because the first filter is now charged by the second filter. Thus, the assumption \( i_1 = 0 \) and Eq. (1) do not hold. Instead, \( i_1 \) is now given by

\[ i_1 = (u_1 - u_2) / R. \]

A correct derivation assuming \( i_2 = 0 \) then yields the model

\[ (RC)^2 \frac{d^2u_2(t)}{dt^2} + 3RC \frac{du_2(t)}{dt} + u_2(t) = u_0(t). \]  \hspace{1cm} (3)

Assume that \( RC = 1 \) sec and calculate by means of the Laplace transform the output \( u_2(t) \) when \( u_0 \) changes as a unit step.

Plot the step response \( u_1(t) \) from 1a) and \( u_2(t) \) from 1b) in the same figure and compare them. Note that the result can be checked by simulations with SIMULINK using "transfer function" blocks.

2. Calculate the response \( u_1(t) \) for the filter in 1a) when the input \( u_0 \) changes sinusoidally as

\[ u_0(t) = \begin{cases} 0, & t < 0 \\ \sin \omega t, & t \geq 0 \end{cases}. \]

The expression for \( u_1(t) \) will contain a transient term, i.e. a decaying term which vanishes as \( t \to \infty \), and a stationary term, which does not vanish as \( t \to \infty \). Express the stationary term in the form \( A \sin(\omega t + \varphi) \), where \( A \) is the amplitude and \( \varphi \) is the so-called phase shift of the stationary signal. This form can be obtained directly by the right choice of inverse Laplace transform.

What are the values of \( A \) and \( \varphi \) when \( \omega \) has the value (i) 0, (ii) 0.1, (iii) 1, (iv) 10, and (v) \( \infty \)? Does the result verify the claim that the filter is a low-pass filter?