Stability analysis of an inverted pendulum

The figure to the right shows an inverted pendulum, whose lower end is attached to a traverse (or a cart) that can move horizontally along a straight line. The mass of the pendulum in the upper end is $m = 1\text{kg}$, its length is $\ell = 1\text{m}$, and the acceleration due to gravity is $g = 10\text{m/s}^2$. The pendulum is attached to the traverse at a horizontal distance $u(t)$ from a reference point, its free end is at the horizontal distance $y(t)$ from the same reference point, and the angle between the pendulum rod and a vertical axis is $\theta(t)$. These variables are functions of time $t$.

1. Show that the pendulum can be modelled by the approximate transfer function

\[ G_p(s) = \frac{Y(s)}{U(s)} = \frac{-10}{s^2 - 10}, \]

where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$, respectively. The following can be assumed: the whole mass of the pendulum is in the upper end, the angle $\theta$ is small, and the acceleration of the mass point in the vertical direction is negligible.

2. Show that the pendulum described by this transfer function is an unstable system.

3. Investigate whether the pendulum can be stabilized with a controller of PID type using the control law $U(s) = G_c(s)(R(s) - Y(s))$, where $G_c(s)$ is the transfer function of the controller and $R(s)$ is the reference value (setpoint) for $Y(s)$. Especially, consider stabilization with a

   a) P controller, $G_c = K_c$;
   
   b) PI controller, $G_c = K_c \left(1 + \frac{1}{T_i s}\right)$;
   
   c) PD controller, $G_c = K_c(1 + T_d s)$.

Which controller(s), if any, can stabilize the pendulum? If there are such controllers, calculate the intervals of the controller parameters ($K_c$, $T_i$, $T_d$) that result in a stable system. Are these intervals affected by the values of the physical quantities $m$, $\ell$ and $g$? If “yes”, how?

4. Above, it was assumed that the controller can adjust the position $u(t)$ directly, but in practice this is not possible. An actuator like, e.g., a stepper motor is needed. Assume that the dynamics of the actuator can be described as a first-order system

\[ G_v(s) = \frac{U(s)}{V(s)} = \frac{1}{Ts + 1}, \]

where $T$ is the time constant of the actuator and $V(s)$ is the controller output. Determine whether the pendulum can be stabilized with the control law $V(s) = G_v(s)(R(s) - Y(s))$ when the controller is any of the controller types in part 3 (i.e., P, PI or PD). Does the time constant $T$ change the stability intervals of the controller parameters? If “yes”, in what way?