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8.0 Overview

So far in this course system properties have been studied in the
- time domain (e.g., step response)
- Laplace domain (e.g., stability)

In this chapter, system properties are studied in the
- frequency domain

by studying the stationary behaviour when the system is excited by a sinusoidal input of given frequency. Stationary behaviour refers to the situation when initial effects have vanished, i.e., when the time \( t \to \infty \).

The output from a linear system will then also change sinusoidally with certain characteristic properties that depend on the system as well as the amplitude and frequency of the input sinusoidal. These properties expressed as function of the input frequency are referred to as the frequency response of the system. System analysis based on the frequency response is called frequency analysis.

The study of sinusoidal inputs is useful because measurement noise and time-varying disturbances can often be approximated by sinusoidal signals.
8. Frequency Response Analysis

8.1 The frequency response for a stable system

In this section, the frequency response for an arbitrary linear, stable, system is derived.

- In order to introduce new concepts step-wise, some simple system elements that essentially lack dynamics are treated first. These elements are often part of more complex systems.

- After this, systems of first and second order are considered. In these systems, dynamics play a prominent role.

- For the derivation of the frequency response of high-order systems, the frequency response of simpler systems is utilized. The decomposition of a high-order system into low-order components connected in series is fundamental in frequency analysis.
8. Frequency Response Analysis

8.1 Frequency response for a stable system

8.1.1 Simple system elements

Static system

A linear static system is described by

\[ y(t) = Ku(t) \]  

(8.1)

where \( u(t) \) is an input, \( y(t) \) is an output, and \( K \) is the system gain. Let the input change sinusoidally as

\[ u(t) = A \sin \omega t \]  

(8.2)

where \( \omega \geq 0 \) is the angular frequency (expressed in radians per time unit) and \( A > 0 \) is the amplitude of the sinusoidal. The output is then

\[ y(t) = KA \sin \omega t \]  

(8.3)

There are now two situations regarding the phase of the output:

- If \( K > 0 \), it is the same as the phase of the input.
- If \( K < 0 \), it is opposite to the phase of the input. In this case, \( y(t) \) has a maximum when \( u(t) \) has a minimum, and vice versa.

Thus, the negative system gain causes a phase shift of \( \pm \pi \) radians, or \( \pm 180^\circ \), in the output.
8.1.1 Simple system elements

It is of interest to rewrite (8.3) so that

- the phase shift is explicitly seen in the equation
- the output amplitude is always a positive quantity

To accomplish this, (8.3) is written as

\[ y(t) = K |A| \sin(\omega t + \varphi) \]  \hspace{1cm} (8.4)

where

\[ \varphi = \begin{cases} 
0 & \text{if } K > 0 \\
-\pi & \text{if } K < 0 
\end{cases} \]  \hspace{1cm} (8.5)

is the phase shift of the system.

The ratio between the amplitudes of the output and input signals is also of interest. In this case, the amplitude ratio is

\[ A_R = |K| \]  \hspace{1cm} (8.6)

Generally, the output can be written as

\[ y(t) = A_R A \sin(\omega t + \varphi) \]  \hspace{1cm} (8.7)

The stationary output of any stable linear system with an input (8.2) can be written as (8.7), where \( A_R \) is the amplitude ratio and \( \varphi \) is the phase shift of the system in question.
**Derivative system**

A system with the transfer function $G(s) = Ks$ has an output that is the time derivative of the input amplified by the factor $K$, i.e.,

$$y(t) = K \frac{du(t)}{dt} \quad (8.8)$$

When the input is sinusoidal as in (8.2), the output becomes

$$y(t) = K \frac{d(A \sin \omega t)}{dt} = KA\omega \cos \omega t \quad (8.9)$$

where the relationship $d(\sin \omega t)/dt = \omega \cos \omega t$ has been used.

Here the output is a cosine function, but by using the trigonometric identity $\cos \omega t = \sin(\omega t + \pi/2)$, (8.9) can be written as

$$y(t) = K\omega A \sin(\omega t + \pi/2) \quad (8.10)$$

Here the phase shift $+\pi/2$ is used, not $-3\pi/2$, although they both give the same output. The reason is not that $+\pi/2$ is closer to 0, but that the derivative yields a *prediction* of the input. Hence, the output can be considered to correspond to a *future* value of the input meaning that the phase of the output is before the phase of the input. Thus, *the derivative is a phase-advancing element*. 
8.1.1 Simple system elements

If the gain $K < 0$, this has to be taken into account as for a static system. Thus, the output is given by (8.7) with

$$A_R = |K|$$  \hspace{1cm} (8.11)

$$\varphi = \begin{cases} 
\pi/2 & \text{if } K > 0 \\
-\pi/2 & \text{if } K < 0
\end{cases}$$  \hspace{1cm} (8.12)

Parallel connection of static and derivative system

A system composed of a parallel connection of a static and a derivative system has the transfer function

$$G(s) = K(1 + Ts)$$  \hspace{1cm} (8.13)

This can be a **PD controller**, but other systems with zeros also have such factors in the numerator of the transfer function.

In the time domain, the output is given by

$$y(t) = Ku(t) + KT \frac{du(t)}{dt}$$  \hspace{1cm} (8.14)

When the input is a sinusoidal as in (8.2), this gives

$$y(t) = KA\sin \omega t + T\omega \cos \omega t$$  \hspace{1cm} (8.15)
The right-hand side of (8.15) can be written as a single sine function by means of the trigonometric identity

\[ \sin \omega t + x \cos \omega t = \sqrt{1 + x^2} \sin(\omega t + \arctan x) \]  

(8.16)

This gives

\[ y(t) = KA \sqrt{1 + (T\omega)^2} \sin(\omega t + \arctan T\omega) \]  

(8.17)

The output is thus given by (8.7) with

\[ A_R = |K| \sqrt{1 + (T\omega)^2} \]  

(8.18)

\[ \varphi = \begin{cases} 
\arctan T\omega & \text{if } K > 0 \\
\arctan T\omega - \pi & \text{if } K < 0 
\end{cases} \]  

(8.19)
8.1 Frequency response for a stable system

8.1.1 Simple system elements

**Integrating system**

A system with the transfer function \( G(s) = Ks^{-1} \) has an output that is the time integral of the input amplified by the factor \( K \). This is equivalent to the time derivative of the output being equal to the input amplified by \( K \), i.e.,

\[
\frac{dy(t)}{dt} = Ku(t) \tag{8.20}
\]

The sinusoidal input (8.2) gives

\[
\frac{dy(t)}{dt} = KA \sin \omega t \tag{8.21}
\]

Because \( \frac{d \cos \omega t}{dt} = -\omega \sin \omega t \), it is easy to verify by differentiation that

\[
y(t) = -KA\omega^{-1} \cos \omega t \tag{8.22}
\]

satisfies (8.20). Because \( \cos \omega t = \sin(\omega t + \pi/2) = -\sin(\omega t - \pi/2) \), substitution into (8.22) yields (8.7) with

\[
A_R = |K|\omega^{-1} \tag{8.23}
\]

\[
\varphi = \begin{cases} 
-\pi/2 & \text{if } K > 0 \\
-3\pi/2 & \text{if } K < 0
\end{cases} \tag{8.24}
\]
8.1.1 Simple system elements

**Parallel connection of static and integrating system**

A system composed of a parallel connection of a static and an integrating system has the transfer function

\[
G(s) = K \left(1 + \frac{1}{Ts}\right) \tag{8.25}
\]

This can, e.g., be the transfer function of a *PI controller*. From (8.3) and (8.22) it is clear that the sinusoidal input (8.2) gives

\[
y(t) = KA \left[\sin \omega t - (T\omega)^{-1}\cos \omega t\right] \tag{8.26}
\]

Taking the trigonometric identity (8.16) and the sign of \( K \) into account yields (8.7) with

\[
A_R = |K| \sqrt{1 + (T\omega)^{-2}} \tag{8.27}
\]

\[
\varphi = \begin{cases} 
\arctan[(T\omega)^{-1}] & \text{if } K > 0 \\
\arctan[(T\omega)^{-1}] - \pi & \text{if } K < 0 
\end{cases} \tag{8.28}
\]
8.1 Frequency response for a stable system

**Time delay**

A time delay of length \( L \) has the transfer function \( G(s) = e^{-Ls} \). In the time domain, the relationship between the output and the input is

\[
y(t) = u(t - L)
\]

(8.29)

The sinusoidal input (8.2) then gives

\[
y(t) = A \sin(\omega(t - L)) = A \sin(\omega t + \varphi)
\]

(8.30)

where

\[
\varphi = -L\omega
\]

(8.31)

As seen from (8.30), a pure time delay has the amplitude ratio

\[
A_R = 1
\]

(8.32)

Note that the phase shift of a time delay is unlimited — the higher the frequency, the more negative the phase shift.
8. Frequency Response Analysis

8.1 Frequency response for a stable system

8.1.2 First-order system

A first-order system is described by the differential equation

\[ T \frac{dy(t)}{dt} + y(t) = Ku(t) \]  

(8.33)

where \( u(t) \) is an input, \( y(t) \) is an output, \( K \) is the gain, and \( T \) is the time constant of the system.

Frequency response via the Laplace domain

As for the previous cases, the frequency response could be derived directly in the time domain. However,

- it is instructive to derive the frequency response via the Laplace domain;
- this will help derive more general relationships for the frequency response,
- which are useful when systems of arbitrary order are considered.
8.1.2 First-order system

Laplace transformation of (8.33) yields

\[ Y(s) = G(s)U(s), \quad G(s) = \frac{K}{Ts+1} \quad (8.34) \]

where \( U(s) \) is the Laplace transform of \( u(t) \), \( Y(s) \) is the Laplace transform of \( y(t) \), and \( G(s) \) is the transfer function of the system. The Laplace transform of the sinusoidal input (8.2) is

\[ U(s) = \frac{A\omega}{s^2+\omega^2} \quad (8.35) \]

which substituted into (8.34) gives

\[ Y(s) = G(s) \frac{A\omega}{s^2+\omega^2} = \frac{K}{Ts+1} \cdot \frac{A\omega}{s^2+\omega^2} \quad (8.36) \]

Since the second-order factor \( s^2 + \omega^2 \) has complex-conjugated zeros, (8.36) has the partial fraction expansion

\[ Y(s) = \frac{B}{Ts+1} + \frac{Cs+D}{s^2+\omega^2} \quad (8.37) \]

where the coefficients \( B, C \) and \( D \) have to be determined so that (8.36) and (8.37) are equivalent.
8.1.2 First-order system

It is useful to consider the inverse Laplace transform of (8.37) before $B$, $C$ and $D$ are determined. The transforms 25, 38 and 39 in the Laplace transform table in Section 4.5 yield

$$y(t) = \frac{B}{T} e^{-t/T} + C \cos \omega t + D \omega^{-1} \sin \omega t$$

(8.38)

We are interested in the stationary solution when $t \to \infty$. Because $T > 0$ for a stable system, and $B$ is finite, the first term on the right-hand side will vanish as $t \to \infty$. Thus, the stationary solution is

$$\bar{y}(t) \equiv \lim_{t \to \infty} y(t) = C \cos \omega t + D \omega^{-1} \sin \omega t$$

(8.39)

This means that the coefficient $B$ does not affect the stationary solution and it is sufficient to determine only $C$ and $D$ if it can be done independently of $B$.

Combination of the first part of (8.36) with (8.37) gives

$$\frac{B}{Ts+1} + \frac{Cs+D}{s^2+\omega^2} = G(s) \frac{A\omega}{s^2+\omega^2}$$

from which

$$Cs + D \equiv G(s)A\omega - \frac{B(s^2+\omega^2)}{Ts+1}$$

(8.40)
8.1.2 First-order system

The identity (8.40) has to apply for arbitrary values of $s$. Choosing $s = j\omega$ means that $s^2 + \omega^2 = 0$. Then (8.40) yields

$$C\omega j + D \equiv G(j\omega)A\omega$$  \hspace{1cm} (8.41)

The identity (8.41) requires that the real part and the imaginary part are satisfied independently. Because $G(j\omega)$ is a complex number with the real part $\text{Re} \ G(j\omega)$ and the imaginary part $\text{Im} \ G(j\omega)$, i.e.,

$$G(j\omega) = \text{Re} \ G(j\omega) + \text{Im} \ G(j\omega) \ j$$

(8.41) yields

$$C = A \ \text{Im} \ G(j\omega) , \quad D = A\omega \ \text{Re} \ G(j\omega)$$  \hspace{1cm} (8.42)

The first-order system (8.34) yields

$$G(j\omega) = \frac{K}{T\omega j + 1} = \frac{K}{T\omega j + 1} \cdot \frac{1 - T\omega j}{1 - T\omega j} = \frac{K - KT\omega j}{1 + (T\omega)^2}$$  \hspace{1cm} (8.43)

from which

$$\text{Re} \ G(j\omega) = \frac{K}{1 + (T\omega)^2} , \quad \text{Im} \ G(j\omega) = \frac{-KT\omega}{1 + (T\omega)^2}$$  \hspace{1cm} (8.44)
8.1.2 First-order system

Substitution of (8.44) into (8.42) and further into (8.39) yields the stationary solution

$$\bar{y}(t) = \frac{KA}{1+(T\omega)^2} \left( \sin \omega t - T\omega \cos \omega t \right)$$

(8.45)

The trigonometrical identity (8.16) applied to (8.45) yields

$$\bar{y}(t) = \frac{KA}{\sqrt{1+(T\omega)^2}} \sin(\omega t - \arctan T\omega)$$

(8.46)

Thus, the stationary solution has the same form as (8.7), i.e.,

$$\bar{y}(t) = A_{R}A \sin(\omega t + \varphi)$$

(8.47)

For a first-order system (8.47) applies with

$$A_{R} = \frac{|K|}{\sqrt{1+(T\omega)^2}}$$

(8.48)

$$\varphi = \begin{cases} -\arctan(T\omega) & \text{if } K > 0 \\ -\arctan(T\omega) - \pi & \text{if } K < 0 \end{cases}$$

(8.49)
8. Frequency Response Analysis

8.1 Frequency response for a stable system

8.1.3 Second-order systems

A second-order system without zeros and with no time delay has the transfer function

\[ G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]  (8.50a)

where \( K \) is the static gain of the system, \( \zeta \) is its relative damping and \( \omega_n > 0 \) is its undamped natural frequency. Since only stable systems are considered, \( \zeta > 0 \). If the poles of the system are real, the transfer function is often written in the form

\[ G(s) = \frac{K}{(T_1s+1)(T_2s+1)} \]  (8.50b)

The parameters of (8.50a) are then given by

\[ \zeta = \frac{T_1+T_2}{2\sqrt{T_1T_2}}, \quad \omega_n = \frac{1}{\sqrt{T_1T_2}} \]  (8.51)

It is here sufficient to consider second-order systems without zeros because systems with zeros can be decomposed into a series connection of two (or more) subsystems and handled by the methods in Section 5.1.5.
8.1 Frequency response for a stable system

**Derivation of frequency response**

Analogously to (8.36), the Laplace transform of the sinusoidal input yields

\[
Y(s) = G(s) \frac{A\omega}{s^2 + \omega^2} = \frac{K\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \cdot \frac{A\omega}{s^2 + \omega^2}
\]  
(8.52)

When \( \zeta > 0 \), there always exists a partial fraction expansion

\[
Y(s) = \frac{B_1 s + B_2}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{Cs + D}{s^2 + \omega^2}
\]  
(8.53)

According to our Laplace transform table, the first term on the right-hand side has an inverse transform containing the factor \( e^{-\zeta \omega_n t} \), where \( t \) denotes time. Because \( \zeta \omega_n > 0 \), \( e^{-\zeta \omega_n t} \rightarrow 0 \) as \( t \rightarrow \infty \) and the term containing this factor will vanish.

This means that also in this case

- the stationary solution to (8.53) is given by (8.39)
- the coefficients \( C \) and \( D \) are given by (8.42)
8.1.3 Second-order systems

In this case

\[ G(j\omega) = \frac{K\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n\omega + \omega_n^2} = \frac{K\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega j} \]

\[ = \frac{K\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega j} \cdot \frac{\omega_n^2 - \omega^2 - 2\zeta\omega_n\omega j}{\omega_n^2 - \omega^2 - 2\zeta\omega_n\omega j} = \frac{K\omega_n^2(\omega_n^2 - \omega^2 - 2\zeta\omega_n\omega j)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \quad (8.54) \]

from which

\[ \text{Re } G(j\omega) = \frac{K\omega_n^2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}, \quad \text{Im } G(j\omega) = \frac{-2K\omega_n^3\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \quad (8.55) \]

Substitution into (8.42) and further into (8.39) gives

\[ \ddot{y}(t) = \frac{K\omega_n^2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} A \left( \sin \omega t - \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \cos \omega t \right) \quad (8.56) \]

where we (for now) assume that \( \omega \neq \omega_n \). Application of the trigonometric identity (8.16) yields the stationary response as

\[ \ddot{y}(t) = \frac{K\omega_n^2\text{sgn}(\omega_n^2 - \omega^2)}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} A \sin(\omega t + \varphi), \quad \varphi = -\arctan \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \quad (8.57) \]
8.1.3 Second-order systems

Eq. (8.57) can be expressed in the form of (8.47) with

\[
A_R = \frac{|K| \omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega_n \omega)^2}} = \frac{|K|}{\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2 \zeta \omega/\omega_n)^2}}
\]  

(8.58)

\[
\varphi = \begin{cases} 
- \arctan \left( \frac{2 \zeta \omega_n \omega}{\omega_n^2 - \omega^2} \right) & \text{if } K \omega_n > K \omega \\
- \arctan \left( \frac{2 \zeta \omega_n \omega}{\omega_n^2 - \omega^2} \right) - \pi & \text{if } K \omega_n < K \omega 
\end{cases}
\]

(8.59a)

\[
= \begin{cases} 
- \arctan \left( \frac{2 \zeta \omega/\omega_n}{1 - (\omega/\omega_n)^2} \right) & \text{if } K \omega_n > K \omega \\
- \arctan \left( \frac{2 \zeta \omega/\omega_n}{1 - (\omega/\omega_n)^2} \right) - \pi & \text{if } K \omega_n < K \omega 
\end{cases}
\]

(8.59b)

For \( \omega = \omega_n \), these reduce to

\[
A_R = \frac{|K|}{2 \zeta}, \quad \varphi = \begin{cases} 
- \pi/2 & \text{if } K > 0 \\
-3\pi/2 & \text{if } K < 0 
\end{cases}, \quad \omega = \omega_n
\]

(8.60–61)

If \( \zeta < 1/\sqrt{2} \approx 0.7 \), the amplitude ratio has a peak (i.e., maximum) at the peak frequency \( \omega_p \), given by

\[
\omega_p = \omega_n \sqrt{1 - 2 \zeta^2} \Rightarrow A_{R, \text{peak}} = \frac{|K|}{2 \zeta \sqrt{1 - 2 \zeta^2}}
\]

(8.62–63)
8.1.4 Systems of arbitrary order

For a stable system of arbitrary order, but no time delay, the response to a sinusoidal input can be expressed by a partial fraction expansion as in (8.37) and (8.53). As before, the stationary solution in the time-domain is given by (8.39) with $C$ and $D$ given by (8.42).

To simplify the notation, we define

$$R(\omega) \equiv \text{Re} \, G(j\omega), \quad I(\omega) = \text{Im} \, G(j\omega)$$  \hspace{1cm} (8.64)

Substitution of (8.42) into (8.39) then gives

$$\ddot{y}(t) = A \left( R(\omega) \sin \omega t + I(\omega) \cos \omega t \right)$$

$$= AR(\omega) \left( \sin \omega t + \frac{I(\omega)}{R(\omega)} \cos \omega t \right)$$  \hspace{1cm} (8.65)

Application of the trigonometrical identity (8.16) yields

$$\ddot{y}(t) = R(\omega) \sqrt{1 + \left( \frac{I(\omega)}{R(\omega)} \right)^2} A \sin(\omega t + \phi)$$

$$= \text{sgn}[R(\omega)] \sqrt{R(\omega)^2 + I(\omega)^2} A \sin(\omega t + \phi)$$  \hspace{1cm} (8.66)

where

$$\phi = \arctan \left[ \frac{I(\omega)}{R(\omega)} \right]$$  \hspace{1cm} (8.67)
Because $G(j\omega)$ is a complex number, it can be characterized by its magnitude $|G(j\omega)|$ and argument $\angle G(j\omega)$, also denoted $\arg G(j\omega)$. According to the theory of complex number,

$$|G(j\omega)| = \sqrt{R(\omega)^2 + I(\omega)^2}, \quad \tan \arg G(j\omega) = \frac{I(\omega)}{R(\omega)}$$  \hspace{1cm} (8.68a,b)$$

The sign of $\text{sgn}[R(\omega)]$ in (8.66) has the same effect on the phase shift as the sign of the gain $K$ in previous sections. Substitution of (8.68) into (8.66) and (8.67) then yields

$$\ddot{y}(t) = |G(j\omega)|A \sin(\omega t + \arg G(j\omega))$$  \hspace{1cm} (8.69)$$

$$\arg G(j\omega) = \begin{cases} \arctan[I(\omega)/R(\omega)] & \text{if } R(\omega) \geq 0 \\ \arctan[I(\omega)/R(\omega)] - \pi & \text{if } R(\omega) < 0 \end{cases}$$  \hspace{1cm} (8.70)$$

From (8.69) it is obvious that

- the magnitude $|G(j\omega)|$ is identical to the amplitude ratio
- the argument $\arg G(j\omega)$ is identical to the phase shift

Thus

$$A_R = |G(j\omega)|, \quad \varphi = \arg G(j\omega)$$  \hspace{1cm} (8.71)$$
Because the function \( \arctan \) only takes values between \(-\pi/2\) and \(+\pi/2\), direct calculation from (8.70) gives an argument in the range

\[-\frac{3}{2} \pi < \arg G(j\omega) < \frac{1}{2} \pi\]  

(8.72)

However, because of the periodicity of the trigonometric functions, (8.68b) would be satisfied for any integer multiple of \(2\pi\) added to \(\arg G(j\omega)\). This means that the solution calculated from (8.70) is ambiguous by an integer multiple of \(2\pi\).

The figure shows a sinusoidal input \(u(t)\) and the stationary output \(y(t)\), which is also sinusoidal. When the axes are normalized as indicated, the values of \(|G|\) and \(\arg G\) can be directly read from the plot for the applied input frequency. Because the sinusoidal signals can be shifted by an integer multiple of \(2\pi\) without any detectable difference, \(\arg G\) is similarly ambiguous also here.

This ambiguity can be solved by the method in Section 5.1.5.
8.1.5 Systems in series

Consider a system composed of \( N \) subsystems connected in series. If the subsystems have the transfer functions \( G_i(s), i = 1, \ldots, N \), the transfer function \( G(s) \) of the full system is given by

\[
G(s) = G_1(s)G_2(s) \cdots G_N(s) = \prod_{i=1}^{N} G_i(s) \tag{8.73}
\]

Conversely, a system with the transfer function

\[
G(s) = K \frac{(T_{n+1}s+1)\cdots(T_{n+m}s+1)}{(T_1s+1)(T_2s+1)\cdots(T_ns+1)} e^{-Ls} \tag{8.74}
\]

can be decomposed into the series-connected subsystems

\[
K, \frac{1}{T_1s+1}, \frac{1}{T_2s+1}, \ldots, \frac{1}{T_ns+1}, (T_{n+1}s+1), \ldots, (T_{n+m}s+1), e^{-Ls}.
\]

If the system has complex-conjugated poles or zeros, they can be included as second-order factors in the decomposition.

Thus, a high-order transfer function can be decomposed into factors of at most second order and a possible time delay.
From (8.73) it follows that the frequency response of the full system is

\[ G(j\omega) = \prod_{i=1}^{N} G_i(j\omega) \]  

(8.75)

where \( G_i(j\omega) \), \( i = 1, \ldots, N \), are the frequency responses of the individual subsystems.

According to the theory of complex numbers, \( G_i(j\omega) \) can be expressed in terms of its magnitude \( |G_i(j\omega)| \) and its argument \( \arg G_i(j\omega) \) as

\[ G_i(j\omega) = |G_i(j\omega)| e^{j\arg G_i(j\omega)} \]  

(8.76)

Substitution into (8.75) yields

\[ G(j\omega) = \prod_{i=1}^{N} |G_i(j\omega)| e^{j\sum_{i=1}^{N} \arg G_i(j\omega)} = \prod_{i=1}^{N} |G_i(j\omega)| \prod_{i=1}^{N} e^{j\arg G_i(j\omega)} \]

\[ = \prod_{i=1}^{N} |G_i(j\omega)| e^{j\sum_{i=1}^{N} \arg G_i(j\omega)} \]  

(8.77)

Naturally, \( G(j\omega) \) can also be expressed as (8.76). From that and (8.77),

\[ |G(j\omega)| = \prod_{i=1}^{N} |G_i(j\omega)| \]  

(8.78)

\[ \arg G(j\omega) = \sum_{i=1}^{N} \arg G_i(j\omega) \]  

(8.79)
Equations (8.78) and (8.79) mean that for the full system the
- amplitude ratio (or magnitude) is obtained as the product of the amplitude ratios (or magnitudes) of the subsystems
- phase shift (or argument) is obtained as the sum of the phase shifts (or arguments) of the subsystems

The user is allowed to decompose the full system into subsystems as desired. However, it is advantageous to decompose into subsystems of first or second order and a possible time delay because we know the frequency responses of such systems. In particular, we know that
- a first-order system has a phase shift in the range \(-\pi/2 < \varphi < 0\)
- a second-order system has a phase shift in the range \(-\pi < \varphi < 0\)

when the gain is positive.

By adding up the phase shifts (arguments) of the individual subsystems according to (8.79), the correct phase shift (argument) for the full system is obtained. This means that a phase shift outside the range of (8.72) can be obtained, i.e., the correct integer multiple of 2\(\pi\) is obtained.
8.1.6 Summary

Table 8.1. Frequency response of low-order systems.

| $G(s)$ | $A_R = |G(j\omega)|$ | $\varphi = \arg G(j\omega)$ |
|--------|---------------------|-----------------------|
| $-1$   | $1$                 | $-\pi$                 |
| $K > 0$| $K$                 | $0$                    |
| $s$    | $\omega$            | $\pi/2$                |
| $1 + Ts$| $\sqrt{1 + (T\omega)^2}$ | $\arctan T\omega$ |
| $1/s$  | $1/\omega$          | $-\pi/2$               |
| $1 + 1/Ts$| $\sqrt{1 + 1/(T\omega)^2}$ | $-\arctan(1/T\omega)$ |
| $e^{-Ls}$| $1$                 | $-L\omega$             |
| $1/Ts + 1$| $\frac{1}{\sqrt{1 + (T\omega)^2}}$ | $-\arctan T\omega$ |
| $\frac{\omega_h^2}{s^2 + 2\zeta\omega_h s + \omega_h^2}$| $\frac{1}{\sqrt{(1 - (\omega/\omega_h)^2)^2 + (2\zeta\omega/\omega_h)^2}}$ | $-\arctan \frac{2\zeta\omega/\omega_h}{1 - (\omega/\omega_h)^2}$, $\omega \leq \omega_h$ |
|                          |                      | $-\pi - \arctan \frac{2\zeta\omega/\omega_h}{1 - (\omega/\omega_h)^2}$, $\omega \geq \omega_h$ |
8. Frequency Response Analysis

8.2 Graphical frequency-response representation

8.2.1 Overview

The complex-valued function $G(j\omega)$ of a system with the transfer function $G(s)$ contains all information about the frequency response of the system—except for a multiple of $2\pi$ in the phase shift.

- $G(j\omega)$ can be considered a system property
- $G(j\omega)$ is called the frequency function

Since $G(j\omega)$ is a complex number, it can be represented in two ways:

$$G(j\omega) = \text{Re} G(j\omega) + j\text{Im} G(j\omega) = |G(j\omega)|e^{j\text{arg}G(j\omega)}$$

- $R(\omega) \equiv \text{Re} G(j\omega)$ is the real part of $G(j\omega)$
- $I(\omega) \equiv \text{Im} G(j\omega)$ is the imaginary part of $G(j\omega)$
- $|G(j\omega)| \equiv \sqrt{R(\omega)^2 + I(\omega)^2}$ is the magnitude of $G(j\omega)$
- $\text{arg}G(j\omega) \equiv \arctan[I(\omega)/R(\omega)] \pm n\pi$ is the argument of $G(j\omega)$

This gives several possibilities of representing $G(j\omega)$ graphically, e.g.,

- Nyquist diagram: $I(\omega)$ vs $R(\omega)$ as $\omega$ varies
- Bode diagram: $|G(j\omega)|$ vs $\omega$ and $\text{arg}G(j\omega)$ vs $\omega$ in two diagrams
8. Frequency Response Analysis

8.2 Graphical representation

8.2.2 Bode diagram

In a Bode diagram, $|G(j\omega)|$ and $\arg G(j\omega)$ are plotted against the frequency $\omega$ in two diagrams.

- The **absolute value** $|G(j\omega)|$ is plotted on a *logarithmic scale*, either expressed as a pure amplitude ratio or with the logarithmic “unit” *decibel* (dB), defined

  $$|G(j\omega)|_{\text{dB}} = 20 \log_{10} |G(j\omega)|$$

  (8.80)

  In this course, the pure amplitude ratio is used.

- The **phase shift** $\arg G(j\omega)$ is plotted on a *linear scale*, either expressed in degrees or radians, defined

  $$1 \text{ rad} = 180^{\circ}/\pi$$

  (8.81)

  In this course, degrees are used in the Bode diagram, but in calculations radians are used.

- The frequency $\omega$ is expressed on a *logarithmic scale* in both diagrams.
8.2 Graphical representation

**First-order system**

The transfer function of a first-order system is \( G(s) = \frac{K}{Ts+1} \). The following expressions have been derived for the amplitude ratio and the phase shift:

- **amplitude ratio**: \( A_R(\omega) = |G(j\omega)| = \frac{K}{\sqrt{1+(\omega T)^2}} \)
- **phase shift**: \( \phi(\omega) = \arg G(j\omega) = -\arctan \omega T \)

The Bode diagram applies to all first-order systems because the normalized amplitude ratio (obtained through division by \( K \)) and a normalized frequency (multiplication by \( T \)) are plotted.

At high frequency:
- slope of \( A_R/K \) is \(-1\)
- \( \phi \to -90^\circ \)
Second-order systems

The transfer function of a second-order system is $G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.

The Bode diagram displays the

- amplitude ratio:
  $$A_R = \frac{K}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

- phase shift:
  $$\varphi = \begin{cases} -\arctan \frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2} & \text{if } K\omega_n \geq K\omega \\ -\arctan \frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2} - \pi & \text{if } K\omega_n \leq K\omega \end{cases}$$

- resonance peak and frequency:
  $$A_R(\omega_p) = \frac{K}{2\zeta\sqrt{1-2\zeta^2}} \text{ at } \omega_p = \omega_n\sqrt{1-2\zeta^2}$$

At high frequency

- slope of $A_R/K$ is $-2$
- $\varphi \rightarrow -180^\circ$

The diagram applies to all second-order systems due to normalized axes.
8.2 Graphical representation

**Time delay**

The transfer function of a time delay is \( G(s) = e^{-Ls} \). The Bode diagram displays the

- **amplitude ratio**: \( A_R(\omega) = 1 \)
- **phase shift**: \( \varphi(\omega) = -\omega L \)

At high frequency

- \( \varphi \to -\infty^\circ \) as \( \omega \to \infty \)

If the frequency axis \( \omega L \) were linear, the slope of the phase shift plot would be \(-1\).
8.2 Graphical representation

8.2.2 Bode diagram

**Numerator time constant**

A time constant in the numerator of a transfer function corresponds to a subsystem with the transfer function \( G(s) = 1 + Ts \). It is also the transfer function of a PD controller (with the gain 1 and derivative time \( T \)). The transfer function has the following characteristics:

- **amplitude ratio:** \( A_R(\omega) = \sqrt{1 + (\omega T)^2} \)
- **phase shift:** \( \varphi(\omega) = \arctan(\omega T) \)

At high frequency:

- slope of \( A_R \) is +1
- \( \varphi \rightarrow +90^\circ \) if \( T > 0 \), \( \varphi \rightarrow -90^\circ \) if \( T < 0 \)

Note the similarity (and the difference) to a first-order system. For the numerator time constant:

- the amplitude plot is symmetrical to the normalized amplitude plot of a first-order system with respect to the line \( A_R = 1 \) (i.e., its “mirror”)
- if \( T > 0 \), the phase plot is symmetrical to the phase plot of a first-order system with respect to the frequency line \( \varphi = 0 \) (i.e., its mirror); if \( T < 0 \), the plots are identical
8.2 Graphical representation

**Integrating system**

An integrator has the transfer function $G(s) = s^{-1}$ with the following characteristics

- amplitude ratio: $A_R(\omega) = 1/\omega$
- phase shift: $\varphi(\omega) = -\pi/2$

At *all* frequencies

- slope of $A_R$ is $-1$
- $\varphi = -90^\circ$

**Derivative system**

A derivative system has the transfer function $G(s) = s$ with the following characteristics

- amplitude ratio: $A_R(\omega) = \omega$
- phase shift: $\varphi(\omega) = +\pi/2$

At *all* frequencies

- slope of $A_R$ is $+1$
- $\varphi = +90^\circ$
8.2 Graphical representation

**Systems in series**

A system composed of \( N \) subsystems in series with the transfer functions \( G_1(s) \), \( G_2(s) \), ..., \( G_N(s) \), has the transfer function

\[
G(s) = \prod_{i=1}^{N} G_i(s)
\]

with the following characteristics

- **amplitude ratio:**
  \[
  A_R(\omega) = \prod_{i=1}^{N} A_{R,i}(\omega)
  \]
  \[
  \log A_R(\omega) = \sum_{i=1}^{N} \log A_{R,i}(\omega)
  \]

- **phase shift:**
  \[
  \varphi(\omega) = \sum_{i=1}^{N} \varphi_i(\omega)
  \]

Because the amplitude plot in a Bode diagram is logarithmic, the logarithmic amplitude plot of the overall system is obtained by adding the logarithmic amplitude plots of the subsystems. This is a useful property if the Bode plot is drawn “by hand”.

Because the phase plot is a Bode diagram is linear, the phase plot of the overall system is obtained by adding the phase plots of the subsystem.
8. Frequency Response Analysis

8.3 Stability analysis of feedback systems

8.3.1 Bode's stability criterion

The system in the block diagram has the following transfer functions:

- $G_p$ is a process to be controlled
- $G_m$ is a measurement device
- $G_v$ is an actuator (e.g., a valve)
- $G_c$ is a controller

The loop transfer function of the system is

$$G_\ell = G_m G_p G_v G_c$$  \hspace{1cm} (8.82)

Consider a system with $G_c = K_c$, $G_v = 1$, $G_p = \frac{e^{-0.1s}}{0.5s+1}$, $G_m = 1$, where the time unit in $G_p$ is “minutes”. The loop transfer of the system is

$$G_\ell = \frac{K_c e^{-0.1s}}{0.5s+1}$$  \hspace{1cm} (8.83)
A thought experiment

- Assume that the setpoint $r$ changes sinusoidally with the angular frequency $\omega = 17 \text{ rad/min}$. At stationary conditions (i.e., when the initial effects from starting the sinusoidal have vanished) the phase shift through the loop transfer is
  \[ \varphi = -0.1 \cdot 17 - \arctan(0.5 \cdot 17) \approx -\pi = -180^\circ \]
  This means that $y_m$ will oscillate exactly in opposite phase to $r$ (and $e$), as illustrated in the figure to the right.

- Assume now that the feedback loop is closed. Because of the minus sign in $e = r - y_m$, $-y_m$ will be exactly in phase with $r$, as shown in the figure to the right. Even if $r$ is set to zero at the same moment the loop is closed, $e$ will continue to oscillate with the same frequency because $e = -y_m$. 

Before loop is closed.

After loop is closed.
8.3.1 Bode’s stability criterion

The amplitude of \( y_m \) can be determined from the amplitude ratio of the system at the frequency 17 rad/min. In this case

\[
A_R(\omega) = \frac{K_c}{\sqrt{1+(\omega T)^2}} \Rightarrow A_R(17) = \frac{K_c}{\sqrt{1+(17 \cdot 0.5)^2}} \approx K_c/8.56 \quad (8.84)
\]

There are now three possibilities.

- If the amplitude of \( y_m \) is smaller than the amplitude of \( r \) and \( e \) before the loop is closed (i.e., if \( K_c < 8.56 \Rightarrow A_R < 1 \)), the amplitude of \( e \) will decrease when the loop is closed. This, in turn, will reduce the amplitude of \( y_m \), making the amplitude of \( e \) even smaller, and so on. Eventually, the oscillation will die out.

- If the amplitude of \( y_m \) is equal to the amplitude of \( r \) and \( e \) before the loop is closed (i.e., if \( K_c = 8.56 \Rightarrow A_R = 1 \)), the amplitude of \( e \) will not change when the loop is closed. This means that the oscillation will continue with the same amplitude “for ever”.

- If the amplitude of \( y_m \) is larger than the amplitude of \( r \) and \( e \) (i.e., if \( K_c > 8.56 \Rightarrow A_R > 1 \)), the amplitude of \( e \) will increase when the loop is closed. This, in turn, will increase the amplitude of \( y_m \), making the amplitude of \( e \) even larger, and so on. The amplitude of the oscillation will become larger and larger and the system is unstable.
8.3 Stability analysis of feedback systems

8.3.1 Bode’s stability criterion

The **stability criterion**

The smallest frequency such that the loop transfer function $G_{\ell}$ has a phase shift equal to $-180^\circ$ is determined. This frequency, $\omega_c$, is the *critical frequency* of the system.

The amplitude ratio of the loop transfer function at this frequency, i.e., $A_{R,\ell}(\omega_c)$, is determined.

The closed-loop system is

- **stable**, if $A_{R,\ell}(\omega_c) < 1$
- **unstable**, if $A_{R,\ell}(\omega_c) > 1$

for the loop transfer function. Note that the stability of the closed-loop system is determined by considering the loop transfer function in open loop.
8.3 Stability analysis of feedback systems

8.3.1 Bode’s stability criterion

How to determine $\omega_c$ and $A_R(\omega_c)$

The critical frequency $\omega_c$ and the amplitude ratio $A_R(\omega_c)$ can be determined in different ways.

- **Graphically**, by drawing the Bode diagram for the loop transfer function $G_\ell$. From such a diagram, it is easy to read off $\omega_c$ at $-180^\circ$ for the phase curve and $A_R(\omega_c)$ from the amplitude curve.

- **Numerically**, by solving the equation

$$\varphi_\ell(\omega_c) \equiv \arg G_\ell(\omega_c) = -\pi \quad (8.85)$$

for $\omega_c$ and calculating

$$A_{R,\ell}(\omega_c) = |G_\ell(\omega_c)| \quad (8.86)$$

It will be shown how $\varphi_\ell(\omega_c) = -\pi$ can be solved iteratively to find $\omega_c$.

- **By simulation** of feedback control using a P controller. The critical frequency $\omega_c$ and the maximum controller gain $K_{c,\text{max}}$ can be determined as described in Section 7.4.1. The amplitude ratio of the loop transfer without a controller is then $A_R(\omega_c) = 1/K_{c,\text{max}}$. 

8.3 Stability analysis of feedback systems

Example 8.3.1

Determine the critical frequency and the amplitude ratio at this frequency for a system with the loop transfer function

\[ G_\ell(s) = G_1(s)G_2(s)G_3(s)G_4(s) \]

where

\[ G_1(s) = e^{-4s}, \quad G_2(s) = \frac{1.5}{2s+1}, \]
\[ G_3(s) = \frac{2}{10s+1}, \quad G_4(s) = \frac{0.8}{5s+1} \]

The solution can be found graphically by means of a Bode diagram. The diagram is constructed by plotting

\[ A_{R,\ell}(\omega) = \frac{1.5}{\sqrt{1+(2\omega)^2}} \cdot \frac{2}{\sqrt{1+(10\omega)^2}} \cdot \frac{0.8}{\sqrt{1+(5\omega)^2}} \]

and

\[ \varphi_\ell(\omega) = -4\omega - \arctan 2\omega - \arctan 10\omega - \arctan 5\omega \]

against the frequency \( \omega \).

From this diagram it is easy to read off \( \omega_c \approx 0.21 \text{ rad/min} \) at \( \varphi_\ell = -\pi = -180^\circ \) and \( A_{R,\ell}(\omega_c) \approx 0.7 \).
Exercise 8.3.1
A process that can be modelled as a pure time delay is controlled by a P controller. The control valve and the measurement device have negligible dynamics, but their gains are \( K_v = 0.5 \) and \( K_m = 0.8 \), respectively. When a small change in the setpoint is made, the controlled process starts to oscillate with constant amplitude and the period 10 min.

a) Which is the controller gain?
b) How large is the time delay?
8.3.2 Stability margins

Gain margin

The gain margin $A_m$ is the factor by which the gain of the loop transfer function has to be changed in order to reach the stability limit. Mathematically,

$$A_m = \frac{1}{A_{R,\ell}(\omega_c)} \quad (8.87)$$

*Stability of the closed-loop system requires* $A_m > 1$.

- The stability margin yields *robustness* not only against variations in the loop gain, but also against variations in other process parameters.

**Example 8.3.2**

Consider a system with the loop transfer function $G_\ell = \frac{K_c e^{-0.1s}}{0.5s+1}$, which was used in the introduction of the Bode stability criterion.

a) Determine a P controller to give the closed-loop system the (designed) gain margin $A_m = 1.7$.

b) Is the closed-loop system stable with the designed P controller if the time delay changes to 0.15 min?
8.3.2 Stability margins

a) It was previously stated that an input sinusoidal with the frequency \( \omega = 17 \text{ rad/min} \) resulted in a phase shift of \(-180^\circ\). Thus, this is the critical frequency \( \omega_c \) of the loop transfer function. We want a P controller such that \( A_m = 1.7 \), i.e., \( A_{R,\ell}(\omega_c) = 1/1.7 \). This is obtained if

\[
A_{R,\ell}(\omega_c) = \frac{K_c}{\sqrt{1+(0.5\cdot17)^2}} = \frac{1}{1.7} \Rightarrow K_c = 5.0
\]

b) To check if the system is stable with \( K_c = 5 \) and \( L = 0.15 \text{ min} \), a Bode diagram could be drawn using these parameters. The diagram gives the new critical frequency and the amplitude ratio at that frequency. Here, the solution is obtained numerically. The phase shift equation is

\[-\pi = -0.15\omega_c - \arctan 0.5\omega_c\]

from which \( \omega_c \) can be calculated iteratively by direct substitution in

\[
\omega_c = (\pi - \arctan 0.5\omega_c)/0.15
\]

This converges to \( \omega_c = 11.6 \text{ rad/min} \) using, e.g., \( \omega_c = 17 \text{ rad/min} \) as starting value. The new gain margin is found by

\[
A_m = \frac{1}{A_{R,\ell}(\omega_c)} = \frac{\sqrt{1+(0.5\cdot11.6)^2}}{5} \approx 1.18 > 1
\]

Because \( A_m > 1 \), the closed-loop system is **stable** with \( L = 0.15 \text{ min} \).
8.3 Stability analysis of feedback systems

**Phase margin**

The phase margin \( \varphi_m \) denotes how much the phase shift at the cross-over frequency of the loop transfer function has to change in order to reach the stability limit. The *cross-over frequency* \( \omega_g \) is the frequency, where the amplitude ratio of the loop transfer function is 1. Mathematically, the phase margin is defined

\[
\varphi_m = \varphi(\omega_g) + \pi
\]  

(8.88)

where \( \omega_g \) is the frequency that satisfies

\[
A_{R,\ell}(\omega_g) = 1
\]  

(8.89)

*Stability of the closed-loop system requires* \( \varphi_m > 0 \).

- The phase margin yields *robustness* not only against variations in the phase shift of the loop transfer, but also against variations in other process parameters.

**Example 8.3.3**

Consider the same system as in Example 8.3.2.

a) Determine a P controller to yield a designed phase margin \( \varphi_m = 30^\circ \).

b) Is the closed-loop system stable if the time delay changes to 0.15 min?
### 8.3.2 Stability margins

**Example 8.3.3**

a) We need to find \( \omega_g \) such that

\[
\varphi(\omega_g) = \varphi_m - \pi = 30^\circ - 180^\circ = -150^\circ = -5\pi/6
\]

\( \omega_g \) can be read off the phase curve of the loop transfer function at \(-150^\circ\). We can also find it numerically by solving

\[
-5\pi/6 = -0.1\omega_g - \arctan 0.5\omega_g
\]

This can be done iteratively by direct substitution in the expression

\[
\omega_g = (5\pi/6 - \arctan 0.5\omega_g)/0.1
\]

The result is \( \omega_g = 12.11 \) rad/min using 11.6 rad/min as starting value. Next, \( K_c \) has to be chosen such that \( A_{R,\ell}(\omega_g) = 1 \), i.e.,

\[
\frac{K_c}{\sqrt{1+(0.5\cdot12.11)^2}} = 1 \quad \Rightarrow \quad K_c = 6.14
\]

b) The stability of the system with \( K_c = 6.14 \) and \( L = 0.15 \) min can be checked in many ways. Here, it is simplest to use the result from Ex. 8.3.2, where \( L = 0.15 \) min gave the critical frequency \( \omega_c = 11.6 \) rad/min. (Note that the phase curve is independent of gains.) Thus,

\[
A_{R,\ell}(\omega_c) = \frac{6.14}{\sqrt{1+(0.5\cdot11.6)^2}} = 1.04 > 1
\]

Since \( A_{R,\ell} > 1 \), the closed-loop system is **unstable** with \( L = 0.15 \) min.
Example 8.3.3

The gain and phase margins can easily be found from a Bode diagram when the controller gain is given. The figure shows the Bode diagram for the loop transfer function \( G_\ell = \frac{K_c e^{-0.1s}}{0.5s + 1} \) with \( K_c = 5 \).

- The gain margin is found as follows. \( \omega_c \) is obtained from the phase curve at \( \varphi_\ell = -180^\circ \). The gain curve yields the amplitude ratio \( A_{R,\ell}(\omega_c) \) at \( \omega_c \).
  \[ A_m = \frac{1}{A_{R,\ell}(\omega_c)} \]

- The phase margin is found as follows. \( \omega_g \) is obtained from the gain curve at \( A_{R,\ell} = 1 \). \( \varphi_m \) is the difference between the phase shift at \( \omega_g \) and \( -180^\circ \).
8.3.3 Numerical solution of frequency relationships

In Example 8.3.2 and 8.3.3 the phase equation was solved by a simple iterative method, where the frequency appearing together with a time delay was solved out. However, there is no guarantee that the solution will converge, and if there is no time delay, the method cannot even be used.

In this section, better methods for solving both the phase equation and the gain equation are developed.

The system can be a general system with the transfer function

\[ G(s) = K \frac{(T_{n+1}s+1) \cdot \ldots \cdot (T_{n}s+1)}{(T_{1}s+1) \cdot \ldots \cdot (T_{n}s+1)} s^m e^{-Ls} \]  

(8.90)

Note that \( m > 0 \) if a factor \( s \) appears in the numerator, and \( m < 0 \) if it is in the denominator. Complex-conjugated poles and zeros will also be considered.

Eq. (8.90) may be any transfer function of interest, but in practice it will be the loop transfer function of the system.
The phase equation

The system (8.90) has the phase shift

\[ \varphi_r = m \frac{\pi}{2} - L \omega - \sum_{i=1}^{n} \arctan T_i \omega + \sum_{i=n+1}^{N} \arctan T_i \omega \]  

(8.91)

The phase shift \( \varphi_r \), expressed in radians, is assumed to be known, and it is desired to find the frequency \( \omega \) satisfying (8.91). The phase shift can have any value relevant for the calculation, but two typical choices are

- \( \varphi_r = -\pi \), if the critical frequency \( \omega = \omega_c \) is to be found
- \( \varphi_r = \varphi_m - \pi \), if the cross-over frequency \( \omega = \omega_g \) is to be found

The following function is defined:

\[ f(\omega) \equiv \varphi_r - m \frac{\pi}{2} + L \omega + \sum_{i=1}^{n} \arctan T_i \omega - \sum_{i=n+1}^{N} \arctan T_i \omega \]  

(8.92)

Since \( f(\omega) = 0 \) at the solution, a possibility is to calculate \( \omega \) iteratively by

\[ \omega_{k+1} = \omega_k - \rho_k f(\omega_k) \]  

(8.93)

where \( \omega_k \) is the solution at iteration step \( k \) and \( \rho_k \) is a parameter that depends on the iteration method.
8.3.3 Numerical solution of frequency relationships

According to the Newton-Raphson method,

\[ \rho_k = \frac{1}{f'(\omega_k)} \]  \hspace{1cm} (8.94)

where \( f'(\omega_k) \equiv \frac{df(\omega_k)}{d\omega_k} \). Differentiation of (8.92) yields

\[ f'(\omega_k) = L + \sum_{i=1}^{n} \frac{T_i}{1+(T_i \omega_k)^2} - \sum_{i=n+1}^{N} \frac{T_i}{1+(T_i \omega_k)^2} \]  \hspace{1cm} (8.95)

An initial guess of the frequency is required to start the iteration. This can be any frequency which might be close to the solution. If no such frequency is known, \( \omega_0 = 0 \) can be used as starting value. However, this will always result in

\[ \rho_0 = \left( L + \sum_{i=1}^{n} T_i - \sum_{i=n+1}^{N} T_i \right)^{-1} \]  \hspace{1cm} (8.96a)

\[ \omega_1 = -\rho_0 (\varphi_r - m\pi/2) \]  \hspace{1cm} (8.96b)

which thus can be used as a better starting value. If desired, the iteration can be continued with \( \rho_k = \rho_0 \), but better convergence is usually achieved if \( \rho_k \) is recalculated at every step, or at least occasionally — the improvement can, in fact, be quite dramatic. If divergence occurs, \( \rho_k \) can be reduced “manually”, e.g., by halving \( \rho_k \).
8.3.3 Numerical solution of frequency relationships

**Complex poles and zeros**

If there are complex poles or zeros, they occur as complex-conjugated pairs. Such a pair has the corresponding complex-conjugated time constants $T_j$ and $T_{j+1}$, which satisfy

$$(T_j s + 1)(T_{j+1} s + 1) = \left(s^2 + 2\zeta \omega_n s + \omega_n^2\right)/\omega_n^2$$

(8.97)

From this is obtained

$$T_j + T_{j+1} = 2\zeta / \omega_n$$

(8.98)

$$\arctan T_j \omega + \arctan T_{j+1} \omega = \arctan \left(\frac{2\zeta \omega_n \omega}{\omega_n^2 - \omega^2}\right)$$

(8.99)

which can substituted into the appropriate positions in (8.92) and (8.96a). In (8.95), the substitution

$$\frac{T_j}{1+(T_j \omega_k)^2} + \frac{T_{j+1}}{1+(T_{j+1} \omega_k)^2} = \frac{2\zeta \omega_n (\omega_n^2 + \omega_k^2)}{\omega_n^4 + 2(2\zeta^2 - 1)\omega_n^2 \omega_k^2 + \omega_k^4}$$

(8.100)

is used. Since the exact value of $\rho_k$ is not very important, an approximation such as

$$\frac{T_j}{1+(T_j \omega_k)^2} + \frac{T_{j+1}}{1+(T_{j+1} \omega_k)^2} \approx \frac{2\zeta (1+(\omega_k/\omega_n)^2)}{\omega_n (1+(\omega_k/\omega_n)^4)}$$

(8.101)

might also be used.
The gain equation

The system (8.90) has the amplitude ratio

\[ A_r = |K| \omega^m \sqrt{\frac{[1+(T_{n+1}\omega)^2] \cdots [1+(T_N\omega)^2]}{[1+(T_1\omega)^2] \cdots [1+(T_n\omega)^2]}} \] \hspace{1cm} (8.102)

The amplitude ratio \( A_r \), expressed as a “pure ratio”, is assumed to be known, and it is desired to find the frequency \( \omega \) satisfying (8.102). The amplitude ratio can have any value relevant for the calculation, but a typical choices is

- \( A_r = 1 \), if the cross-over frequency \( \omega = \omega_g \) is to be found

The following function is defined:

\[ g(\omega) \equiv A_r - |K| \omega^m \sqrt{\frac{[1+(T_{n+1}\omega)^2] \cdots [1+(T_N\omega)^2]}{[1+(T_1\omega)^2] \cdots [1+(T_n\omega)^2]}} \] \hspace{1cm} (8.103)

Since \( g(\omega) = 0 \) at the solution, \( \omega \) can be calculated iteratively by

\[ \omega_{k+1} = \omega_k - \sigma_k g(\omega_k) \] \hspace{1cm} (8.104)

where \( \omega_k \) is the solution at iteration step \( k \) and \( \sigma_k \) is a parameter that depends on the iteration method.
8.3.3 Numerical solution of frequency relationships

According to the Newton-Raphson method,

\[ \sigma_k = \frac{1}{g'(\omega_k)} \]  

(8.105)

where \( g'(\omega_k) \equiv \frac{d g(\omega_k)}{d \omega_k} \). Differentiation of (8.103) yields

\[ g'(\omega_k) = \left[ -m + \sum_{i=1}^{n} \frac{(T_i \omega_k)^2}{1+(T_i \omega_k)^2} - \sum_{i=n+1}^{N} \frac{(T_i \omega_k)^2}{1+(T_i \omega_k)^2} \right] \frac{A_r - g(\omega_k)}{\omega_k} \]  

(8.106)

An initial guess of the frequency is required to start the iteration. This can be any frequency which might be close to the solution. However, \( \omega_0 = 0 \) can not be used as a starting value. The critical frequency \( \omega_c \), which is often known, would usually be a good starting value.

If desired, the iteration can be continued with \( \sigma_k = \sigma_0 \), but better convergence is probably achieved if \( \sigma_k \) is recalculated at every step, or at least occasionally. If divergence occurs, \( \sigma_k \) can be reduced “manually”, e.g., by halving \( \sigma_k \).
8.3.3 Numerical solution of frequency relationships

**Complex poles and zeros**

If there are complex poles or zeros, they occur as complex-conjugated pairs. If the corresponding complex-conjugated time constants are $T_j$ and $T_{j+1}$,

\[
[1 + (T_j \omega)^2][1 + (T_{j+1} \omega)^2] = 1 + 2(2\zeta^2 - 1) \left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4
\]

is substituted into (8.103). In (8.106), the substitution

\[
\frac{(T_j \omega_k)^2}{1+(T_j \omega_k)^2} + \frac{(T_{j+1} \omega_k)^2}{1+(T_{j+1} \omega_k)^2} = \frac{2\omega_k^2[2\zeta^2-1]\omega_n^2 + \omega_k^2}{\omega_n^4 + 2(2\zeta^2-1)\omega_n^2 \omega_k^2 + \omega_k^4}
\]

is used. Since the exact value of $\sigma_k$ is not very important, an approximation such as

\[
\frac{(T_j \omega_k)^2}{1+(T_j \omega_k)^2} + \frac{(T_{j+1} \omega_k)^2}{1+(T_{j+1} \omega_k)^2} \approx \frac{2\omega_k^4}{\omega_n^4 + \omega_k^4}
\]

might also be used.
Exercise 8.3.2

Calculate $K_{c,\text{max}}$ for the system below using frequency analysis.

$$G_m = \frac{1}{s+1}, \quad G_p = \frac{1}{5s+1}, \quad G_v = \frac{1}{2s+1}, \quad G_c = K_c$$
8. Frequency Response Analysis

8.4 Controller design in the frequency domain

In this section it is shown how PI, PD and PID controllers can be designed to satisfy stability and performance criteria in the frequency domain.

- The used stability criteria are the gain margin $A_m$ and the phase margin $\varphi_m$. Usually, $A_m \approx 2$ and $\varphi_m \approx 45^\circ (\pi/4)$ are good values.

- The cross-over frequency $\omega_g$ is related to performance — the higher the cross-over frequency, the better the performance. Usually, a good value is $\omega_g \approx 0.3\omega_c$, where $\omega_c$ is the critical frequency of the uncontrolled (or P-controlled) system.

8.4.1 Design of PI controllers

A PI controller has the transfer function

$$G_{\text{PI}}(s) = K_c \left(1 + \frac{1}{T_i s}\right) = K_c \frac{T_i s + 1}{T_i s} \quad (8.110)$$

If the system to be controlled has the transfer function $G(s)$, the loop transfer function is

$$G_\ell(s) = G(s)G_{\text{PI}}(s) = G(s)K_c \frac{T_i s + 1}{T_i s} \quad (8.111)$$
8.4 Controller design in the frequency domain

The amplitude ratio and the phase shift of the loop transfer function are

\[
A_{R,\ell}(\omega) = |G(j\omega)| \frac{K_c}{T_i \omega} \sqrt{1 + (T_i \omega)^2}
\]  

(8.112)

\[
\varphi_{\ell}(\omega) = \arg G(j\omega) + \arctan T_i \omega - \frac{\pi}{2}
\]  

(8.113)

**Design for desired phase margin**

The integral time \( T_i \approx \frac{5}{\omega_g} \), where \( \omega_g \) is the cross-over frequency, is usually a good choice for a PI controller. Based on this choice, a PI controller for a desired phase margin \( \varphi_m \) can be designed as follows.

1. Solve (8.113) for \( \omega = \omega_g \) with \( \varphi_{\ell} = -\pi + \varphi_m \) and \( T_i \omega_g = 5 \), i.e.,

\[
\varphi_m - \frac{\pi}{2} - \arctan 5 - \arg G(j\omega_g) = 0
\]  

(8.114)

2. Solve (8.112) for \( K_c \) with \( A_{R,\ell} = 1 \), \( \omega = \omega_g \), and \( T_i \omega_g = 5 \), i.e.,

\[
K_c = \frac{5}{\sqrt{26}} |G(j\omega_g)|^{-1}
\]  

(8.115)

3. The integral time is \( T_i = \frac{5}{\omega_g} \).
Example 8.4.1

Design a PI controller for a system with the transfer function

\[ G(s) = \frac{e^{-s}}{10s+1} \]

to achieve the phase margin a) \( \varphi_m = 30^\circ \), b) \( \varphi_m = 60^\circ \). Also calculate controller tunings by the classical methods in Section 7.4 and 7.5.

a) Eq. (8.114) with the pertinent expression for \( \arg G(j\omega_g) \) (see Section 8.1.6) has to be solved with \( \varphi_m = 30^\circ = \pi/6 \). This can be done iteratively according to (see Section 8.3.3)

\[ \omega_{k+1} = \omega_k - \rho_k f(\omega_k) \]

where \( f(\omega_k) = -\pi/3 - \arctan 5 + \omega_k + \arctan 10\omega_k \)

\[ \rho_k = \left(1 + \frac{10}{1+100\omega_k^2}\right)^{-1} \]

Using \( \rho_0 = 1/11 \) and \( \omega_1 = 7\pi/(11 \cdot 9) \approx 0.22 \) as starting values yields \( \rho_1 = 0.37, \omega_2 = 0.61, \rho_2 = 0.79, \omega_3 = 0.93, \rho_3 = 0.90, \omega_4 = 0.9542 = \omega_g \).

The controller gain is \( K_c = 5\sqrt{1 + (10 \cdot 0.9542)^2}/\sqrt{26} \approx 9.41 \) and the integral time is \( T_i = 5/\omega_g = 5/0.9542 = 5.24 \).
8.4.1 Design of PI controllers

Example 8.4.1

b) \( \varphi_m = 60^\circ = \pi / 3 \) means that

\[
f(\omega_k) = -\pi / 6 - \arctan 5 + \omega_k + \arctan 10 \omega_k
\]

A larger phase margin means a lower cross-over frequency. Based on a), we might therefore choose \( \omega_0 = 0.9 \) as starting value for the iteration. This gives \( \rho_0 = 0.9 \), which will be used throughout the calculation to get \( \omega_1 = 0.48 \), \( \omega_2 = 0.53 \), \( \omega_3 = 0.514 \), \( \omega_4 = 0.5179 \), \( \omega_4 = 0.5170 \), \( \omega_6 = 0.5172 = \omega_g \).

The controller gain is \( K_c = 5 \sqrt{1 + (10 \cdot 0.5172)^2 / \sqrt{26}} \approx 5.17 \) and the integral time is \( T_i = 5 / \omega_g = 5 / 0.5172 \approx 9.67 \).

The result of all considered methods is summarized below.

<table>
<thead>
<tr>
<th>PI Param.</th>
<th>( \varphi_m ) 30°</th>
<th>Z-N</th>
<th>ITAE Regul.</th>
<th>CHR 20% Regul.</th>
<th>CHR 0% Regul.</th>
<th>( \varphi_m ) 60°</th>
<th>ITAE Track.</th>
<th>CHR 20% Track.</th>
<th>CHR 0% Track.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_c )</td>
<td>9.41</td>
<td>9.00</td>
<td>8.15</td>
<td>7.00</td>
<td>6.00</td>
<td>5.17</td>
<td>4.83</td>
<td>3.50</td>
<td>6.00</td>
</tr>
<tr>
<td>( T_i )</td>
<td>5.24</td>
<td>3.33</td>
<td>3.10</td>
<td>2.30</td>
<td>4.00</td>
<td>9.67</td>
<td>9.87</td>
<td>12.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>
8.4.1 Design of PI controllers

Example 8.4.1

The figure shows simulated setpoint responses for
a) $\varphi_m = 30^\circ$ (full line),
b) $\varphi_m = 60^\circ$ (dashed line).
8.4.2 Design of PD controllers

A PI controller can be designed to yield a desired phase margin, but it might require a low cross-over frequency, which means that the performance might not be good enough. A PD controller can be designed for a desired phase margin and a desired cross-over frequency.

A PD controller with a derivative filter has the transfer function

\[ G_{PDf}(s) = K_c \left( 1 + \frac{T_d s}{T_f s + 1} \right) = K_c \frac{(T_d + T_f) s + 1}{T_f s + 1} \]  

(8.116)

where \( T_f \) is the filter time constant. The PD controller causes a phase shift

\[ \varphi_{PDf} = \arctan(T_d + T_f) \omega - \arctan T_f \omega = \arctan \left( \frac{T_d \omega}{1 + (T_d + T_f)T_f \omega^2} \right) \]  

(8.117)

For positive \( T_d \) and \( T_f \), this phase shift is positive. The maximum phase lift is obtained at the frequency \( \omega = \omega_{max} \), where

\[ \omega_{max} = \left[ (T_d + T_f)T_f \right]^{-1/2} \]  

(8.118)

This gives the phase lift

\[ \varphi_{max} = \arctan(0.5T_d \omega_{max}) \]  

(8.119)
8.4 Controller design in the frequency domain 8.4.2 PD controller

**Design for desired phase margin and cross-over frequency**

It is the derivative of the PD controller that produces the phase lift. The phase lift should be enough to satisfy the phase margin requirement, but excessive derivative action is not desired. This means that the maximum phase lift should be at the cross-over frequency, i.e., $\omega_{\text{max}} = \omega_g$.

If the system to be controlled has the transfer function $G(s)$, the loop transfer function is

$$G_\ell(s) = G(s)G_{\text{PDf}}(s) = G(s)K_c \frac{(T_d + T_f)s + 1}{T_f s + 1}$$  \hspace{1cm} (8.120)

With the maximum phase lift at the cross-over frequency, (8.118) and (8.119) with $\omega_{\text{max}} = \omega_g$ apply. The amplitude ratio and phase shift equations at the cross-over frequency then become

$$|G(j\omega_g)| \frac{K_c}{T_f \omega_g} = 1$$  \hspace{1cm} (8.121)

$$\phi_m - \pi - \arg G(j\omega_g) - \arctan(0.5T_d \omega_g) = 0$$  \hspace{1cm} (8.122)
8.4.2 PD controller

A PD controller for desired phase margin and cross-over frequency can now be designed as follows.

1. Calculate the derivative time from (8.122), i.e.,

\[ T_d = \frac{2}{\omega_g} \tan[\varphi_m - \pi - \arg G(j\omega)] \quad (8.123) \]

2. Calculate the derivative filter time constant from (8.118), i.e.,

\[ T_f = -0.5T_d + \sqrt{0.25T_d^2 + \omega_g^{-2}} \quad (8.124) \]

3. Calculate the controller gain from (8.121), i.e.,

\[ K_c = T_f \omega_g \left| G(j\omega) \right|^{-1} \quad (8.125) \]
Example 8.4.2

Design a PD controller with a derivative filter for a system with the transfer function

\[ G(s) = \frac{e^{-s}}{10s+1} \]

to achieve the phase margin \( \varphi_m = 60^\circ \) and the cross-over frequency 1 rad/time unit.

Eq. (8.123) gives

\[
T_d = \frac{2}{\omega_g} \tan \left[ \frac{1}{3} \pi - \pi + \omega_g + \arctan 10\omega_g \right]
\]

\[
= 2 \tan \left[ -\frac{2}{3} \pi + 1 + \arctan 10 \right] = 0.79
\]

Eq. (8.124) gives

\[
T_f = -0.5T_d + \sqrt{0.25T_d^2 + \omega_g^{-2}}
\]

\[
= -0.5 \cdot 0.79 + \sqrt{0.25 \cdot 0.79^2 + 1} = 0.68
\]

Eq. (8.125) gives

\[
K_c = T_f \omega_g \sqrt{1 + (10\omega_g)^2} = 0.68 \sqrt{101} = 6.83
\]
8.4.3 Design of PID controllers

A PD controller can be designed for stability (phase margin) and performance (cross-over frequency), but a drawback is that the steady-state control error will not be zero due to the lack of integral action. This can be remedied by connecting a PI and a PD controller in series. However, the previous PI and PD controller designs are not necessarily optimal for that purpose.

A PID controller on series form with a derivative filter has the transfer function

\[
G_{\text{PIPDf}}(s) = K'_c \left(1 + \frac{1}{T'_i s}\right) \left(1 + \frac{T'_d s}{T'_f s + 1}\right) = K'_c \frac{(T'_i s+1)((T'_d+T'_f)s+1)}{T'_i s(T'_f s + 1)} \tag{8.126}
\]

where “primed” parameter symbols are used to distinguish them from the corresponding parameters in the parallel form of the PID controller. The amplitude ratio and the phase shift equations of (8.126) are

\[
A_{R,\text{PIPDf}}(\omega) = \frac{K'_c}{T'_i \omega} \left[\frac{[1+(T'_i \omega)^2][1+(T'_d+T'_f)^2 \omega^2]}{[1+(T'_f \omega)^2]}\right]^{1/2} \tag{8.127}
\]

\[
\varphi_{\text{PIPDf}}(\omega) = -\frac{\pi}{2} + \arctan T'_i \omega + \arctan[(T'_d + T'_f)\omega] - \arctan T'_f \omega \tag{8.128}
\]
Design for desired phase margin and cross-over frequency

As for the PD controller design, it is reasonable to choose the maximum phase lift of the PD part of the PID controller to occur at the cross-over frequency. From (8.118) it follows that this is achieved if

\[(T_d' + T_f')T_f'\omega_g^2 = 1\]  \hspace{1cm} (8.129)

Similarly to (8.121) and (8.122), when the integral part is added, the amplitude ratio and phase shift equations for the loop transfer function at the cross-over frequency become

\[|G(j\omega_g)|\frac{K'_c[1+(T'_i\omega_g)^2]^{1/2}}{T'_i T'_f \omega_g^2} = 1\]  \hspace{1cm} (8.130)

\[\varphi_m - 0.5\pi - \arg G(j\omega_g) - \arctan\frac{(T'_i+0.5T'_d)\omega_g}{1-0.5T'_i T'_d\omega_g^2} = 0\]  \hspace{1cm} (8.131)

In (8.131), \(\arctan T'_i\omega_g\) and \(\arctan(0.5T'_d\omega_g)\) have been combined. Solution of (8.131) for the integral and derivative times gives

\[\frac{(T'_i+0.5T'_d)\omega_g}{1-0.5T'_i T'_d\omega_g^2} = \tan[\varphi_m - 0.5\pi - \arg G(j\omega_g)]\]  \hspace{1cm} (8.132)

The solution of \(T_d'\) from (8.132) depends on how \(T'_i\) is specified.
8.4.3 PID controller

Desired phase margin and cross-over frequency

**$T_i'$ or $T_i'\omega_g$ is known**

If $T_i'\omega_g$ is known, $T_i'\omega_g \approx 5$ is a typical value.

A PID controller for desired phase margin and cross-over frequency can now be designed as follows.

1. Calculate the derivative time from (8.132), which can be rewritten as

   $$T_d' = \frac{2}{\omega_g} \tan[\varphi_m - 0.5\pi - \arctan(T_i'\omega_g - \arg G(j\omega_g))] \quad (8.133)$$

2. Calculate the derivative filter time constant from (8.129), i.e.,

   $$T_f' = -0.5T_d' + \sqrt{0.25T_d'^2 + \omega_g^{-2}} \quad (8.134)$$

3. Calculate the controller gain from (8.130), i.e.,

   $$K_c' = \frac{T_i'T_f'\omega_g^2}{[1+(T_i'\omega_g)^2]^{1/2}} \left|G(j\omega_g)\right|^{-1} \quad (8.135)$$

**The ratio $T_i'/T_d'$ is known**

If $T_i'/T_d'$ is known, $T_i'/T_d' \approx 4$ is a typical value. Eq. (8.133) is replaced by

$$T_d' = \frac{\omega_g}{1+0.5T_d'/T_i'} \left[-p + \sqrt{p^2 + \frac{2T_d'T_i'}{T_i'}}\right], \quad p = \frac{1+0.5T_d'/T_i'}{\tan[\varphi_m-0.5\pi-\arg G(j\omega_g)]} \quad (8.136)$$
Example 8.4.3

Design a PID controller with a derivative filter for a system with the transfer function

\[ G(s) = \frac{e^{-s}}{10s + 1} \]

to achieve the phase margin \( \phi_m = 60^\circ \) and the cross-over frequency 1 rad/time unit. Use \( T'_i \omega_g = 5 \).

Eq. (8.133) gives

\[
T'_d = \frac{2}{\omega_g} \tan \left[ \frac{1}{3} \pi - \frac{1}{2} \pi - \arctan 5 + \omega_g + \arctan 10 \omega_g \right]
\]

\[ = 2 \tan \left[ -\frac{1}{6} \pi - \arctan 5 + 1 + \arctan 10 \right] = 1.29 \]

Eq. (8.134) gives

\[
T'_f = -0.5T'_d + \sqrt{0.25T'_d^2 + \omega_g^{-2}}
\]

\[ = -0.5 \cdot 1.29 + \sqrt{0.25 \cdot 1.79^2 + 1} = 0.54 \]

Eq. (8.135) gives

\[
K'_c = T'_iT'_f \omega_g^2 \sqrt{\frac{1+(10\omega_g)^2}{1+(T'_i \omega_g)^2}} = 5 \cdot 0.54 \sqrt{\frac{101}{26}} = 5.36 \]
Exercise 8.4.1

Design a PID controller with a derivative filter for a system with the transfer function

$$G(s) = \frac{4}{(4s+1)^3}$$

to achieve the phase margin $\phi_m = 35^\circ$ and the cross-over frequency $2 \text{ rad/time unit}$. Use $T_i' \omega_g = 5$. 