6. Stability

As shown in the first two chapters, a successful controller design requires compromises between performance ("speed") and stability.

- An uncontrolled system that is stable may become unstable by aggressive control.
- On the other hand, there are also systems that are unstable without control which can be stabilized by control.

We can conclude that stability is necessary, but not sufficient, for good control.

It is obvious that we need systematic methods to determine if a system — controlled or uncontrolled — is stable or unstable.
6. Stability

6.1. Stability definitions

Stability can be defined in several different ways.

− For all practical purposes, the different definitions are equivalent for linear systems.
− In a given situation, a certain definition may be more convenient to use than other definitions.

Therefore, it is useful here to mention the most common stability definitions.

The two following, quite concrete definitions, are general in so far as that they

▪ apply for linear and nonlinear systems
▪ independent of the type of system description (transfer function or state-space model).
6.1. Stability definitions

6.1.1 Asymptotic stability

A system is asymptotically stable if it returns to its initial state after a transient disturbance.

- A typical transient disturbance is a pulse, and in practice, many calculations become easier if we assume that the pulse is an impulse.
- A step change is not a transient disturbance.

Remark 1. Asymptotic stability is often defined in more mathematical terms than the above, which means that the definitions seem to be different. They are, however, equivalent.

6.1.2 Input-output stability

A system is input-output stable if a limited input signal results in a limited output signal.

- A typical limited input signal is a step change.

Remark 2. It follows from the definition that an input-output stable system has a finite gain at all “frequencies” (see chapter 8).
6. Stability

6.2. Poles and stability

To be useful in the mathematical analysis and design, the stability definitions have to be formulated in more mathematical terms. We will here

- consider the time response (the transient response) of an arbitrary system (without time delay)
- when it is subject to (i) a transient, (ii) a permanent, change of the input signal.

6.1.2 Time response of a system

In accordance with Section 4.3 and equation (4.27), the transfer function of a system without a time delay can generally be written

\[
G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\]

where

\[
A(s) \equiv s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n
\]

is the characteristic polynomial of the system.
6.1.2 Time response of a system

Assume that the characteristic polynomial can be *factorized* such that

\[
A(s) = (s - p_1)(s - p_2) \cdots (s - p_n)
\]  

(6.3)

where \( p_k, k = 1, \ldots, n \), are the *zeros* of the polynomial, which are also the *poles* of the system.

If we initially assume that the *poles are real and distinct*, and that the system is strictly proper (i.e., \( m < n \)), there exists a partial fraction expansion

\[
G(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + \frac{C_n}{s - p_n}
\]  

(6.4)

where the constants \( C_k, k = 1, \ldots, n \), can be determined as described in section 4.4.2.

The output signal \( Y(s) \) of the system is then given by

\[
Y(s) = \left( \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + \frac{C_n}{s - p_n} \right) U(s)
\]  

(6.5)

where \( U(s) \) is its input signal.
6.1.2 Time response of a system

Assume that the input signal is an impulse, i.e., a transient disturbance as in the definition of asymptotic stability.

The Laplace transform of the impulse is $U(s) = I$. Substitution into (6.5) and the inverse Laplace transform give

$$y(t) = C_1e^{p_1t} + C_2e^{p_2t} + ... + C_ne^{p_nt} , \quad t \geq 0$$ (6.6)

The condition for asymptotic stability is that $y(t) \to 0$ when $t \to \infty$. We see that this is fulfilled if and only if all $p_k < 0$, $k = 1,\ldots, n$.

Assume now that the input signal is a step change, i.e., a permanent disturbance as in the definition of input-output stability.

If the step has the magnitude $u_{\text{step}}$, the input has the Laplace transform $U(s) = u_{\text{step}} / s$.

Substitution into (6.5) and the inverse Laplace transform give, if $p_k \neq 0$,

$$y(t) = C_1u_{\text{step}} \left(1 - e^{p_1t}\right) + C_2u_{\text{step}} \left(1 - e^{p_2t}\right) + ... + C_nu_{\text{step}} \left(1 - e^{p_nt}\right) , \quad t \geq 0$$ (6.7)

The output signal is limited if and only if all $e^{p_k t}$, $k = 1,\ldots, n$, are limited for $t \geq 0$. As above, this applies if and only if all $p_k < 0$, $k = 1,\ldots, n$.

If $p_k = 0$, the inverse transform gives another solution, where $y(t)$ grows with time $t$. 
6.1.2 Time response of a system

**Complex zeros** of the characteristic polynomial occur as *complex conjugate pairs*. In a partial fraction expansion, we can choose between

- combining such pairs into a factor of second order (see section 4.4.2)
- calculating with complex numbers (see below)

Assume that $p_1 = \sigma + j\omega$ and $p_2 = \sigma - j\omega$. The first two terms on the right-hand side in (6.6) give

$$y_{1+2}(t) = C_1e^{(\sigma+j\omega)t} + C_2e^{(\sigma-j\omega)t} = Ie^{\sigma t} \left(C_1e^{j\omega t} + C_2e^{-j\omega t}\right)$$

$$= Ie^{\sigma t} \left((C_1 + C_2)\cos(\omega t) + j(C_1 - C_2)\sin(\omega t)\right)$$

where the last equality follows from Euler’s formula.

- Since the signal $y_{1+2}(t)$ must be real, it follows that $C_1$ and $C_2$ are complex conjugates. The right-hand side in (6.8) is then also real.

- Since the trigonometric functions in (6.8) are limited (finite), $y_{1+2}(t) \to 0$ when $t \to \infty$ if and only if $\sigma < 0$, i.e., $\text{Re}(p_k) < 0$.

- The same conditions give a limited output signal when the input signal is a permanent disturbance, such as a step change (possibly added to a sine oscillation of a given frequency).
6.1.2 Time response of a system

If the characteristic polynomial contains **multiple zeros**, we get a

- partial fraction expansion whose inverse transform, besides similar terms as in the expressions above, also contains **products of exponential functions** and the time \( t \) raised to a certain power.

- Since the exponential function \( e^{p_k t} \) with \( \text{Re}(p_k) < 0 \) decreases faster than \( t^n \) grows, such terms will then tend towards zero when \( t \rightarrow \infty \).

Hence, it follows that **the stability conditions given above apply also when the system has multiple poles.**
6.2 Poles and stability

6.2.2 Stability conditions expressed in terms of system poles

According to the analysis above the stability condition can be expressed by the system poles as follows:

A continuous-time system is *stable if and only if all system poles* $p_k$, $k = 1, \ldots, n$, *are located in the left half of the complex plane*, i.e., if

$$\text{Re}(p_k) < 0, \quad k = 1, \ldots, n.$$ (6.9)

The system poles are the zeros of the characteristic equation $A(s) = 0$.

**Remark 3.** For *linear systems*, stability is a *system property*, i.e.,

- if the stability condition is fulfilled for *any* transient or bounded input signal, then the stability condition is fulfilled for *all* such input signals.

This need not be the case for the nonlinear systems.
6.2 Poles and stability

6.2.3 Feedback systems

Obviously, the above result also applies to feedback (controlled) systems.

In a simple control system, there are components with the following transfer functions:

- $G_p$ for the process that will be controlled
- $G_c$ for a controller
- $G_m$ for a measuring device

![Feedback control loop diagram]

**Figure 6.1.** Feedback control loop.

Block diagram algebra gives

$$Y = \frac{G_p G_c}{1 + G_p G_c G_m} R + \frac{1}{1 + G_p G_c G_m} V.$$  \hspace{1cm} (6.10)

Here we have

- the loop transfer function $G_k = G_p G_c G_m$  \hspace{1cm} (6.11)
- the characteristic equation $1 + G_k = 0$  \hspace{1cm} (6.12)
6.2 Poles and stability

To solve (6.12), some rearrangement is useful. Let us define

\[ G_k(s) = \frac{B_k(s)}{A_k(s)}, \]  

which substituted into (6.12) gives the characteristic equation in the form

\[ A(s) = A_k(s) + B_k(s) = 0. \]  

Sometimes it happens that \( G_p, G_c \) or \( G_m \) has a factor \( s - p \) in the denominator which is also found in the numerator of another transfer function. Such a factor shall not be cancelled out from \( G_k \)! This is especially important when \( \text{Re}(p) < 0 \) (i.e. an unstable system).

The process often contains a time delay, and when the system is a feedback system, this time delay occurs in \( G_k(s) \). For a stability analysis according to Routh-Hurwitz’s test in section 6.3.1, it is then necessary to approximate the time delay with a rational expression (see section 5.4), which allows \( B_k(s) \) and \( A_k(s) \) to be expressed as pure polynomials. Then, the stability analysis is only approximate.
6.2 Poles and stability

**Exercise 6.1**

Show that the system \( G_p(s) = \frac{10}{s-1} \) is unstable. Investigate if it can be stabilized by a P-controller.

**Exercise 6.2**

Is the system \( G_p(s) = \frac{1}{s^2 + 2s + 2} \) stable or unstable? Investigate if the closed-loop system is stable when the system is controlled by a PI-controller with

(a) \( K_c = 1 \), \( T_i = 0.5 \);

(b) \( K_c = 15 \), \( T_i = 0.5 \);

(c) \( K_c = 15 \), \( T_i = 0.25 \).
6. Stability

6.3. Analysis methods

The use of the stability criterion, defined by the system poles, requires that the poles can be determined.

- For systems of higher order than 2 it can be difficult or even impossible to determine the poles analytically, but if all the parameters are given, the poles can be calculated numerically.

- Often it is of interest to investigate the stability limits as a function of one or several undetermined parameters (for example, controller parameters), and preferably so that the limits are obtained as analytical expressions. This is problematic for high-order system.

- Another complication arises if the system contains a time delay that is included in the characteristic equation. This situation arises if a feedback-controlled system has a time delay. The calculation of the system poles then requires that the time delay is approximated with a rational expression, which means that the poles can only be determined approximately.

For these reasons, a number of methods for stability analysis have been developed, which give analytical expressions or, in principle, exact (numerical) solutions for systems with a time delay. The following methods are studied in this course:
6.3. Analysis methods

1. **Bode’s stability criterion**, which is studied in section 8.4. This is a so-called frequency-domain method, which can handle time delays without approximation. The analysis can be done “graphically” or numerically.

2. **Nyquist’s stability criterion**, which is studied in section 8.4, but only superficially. This is a more general variation of Bode’s stability criterion. In this case the analysis can also be done graphically or numerically.

3. **Routh-Hurwitz’s stability criterion**, which is studied in section 6.3.1. This method can give stability intervals with respect to various parameters, for example, controller parameters. High system orders cause no special problems, but time delays can not be handled accurately.

4. **Stability analysis by “direct substitution”**, which is studied in section 6.3.2. This method uses the fact that the system poles, i.e. the zeros of the characteristic equation, must be located on the imaginary axis of the complex plane at the stability limit. Time delays can be handled accurately, but the calculations for systems of high order tend to become difficult.
6.3. Analysis methods

6.3.1 Routh-Hurwitz's stability criterion

The Routh-Hurwitz stability criterion requires that the characteristic equation can be written as a polynomial,

\[ A(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0, \]  

(6.15)

where the coefficient \( a_0 \) is included here.

As noted, a possible time delay (\( e^{-Ls} \)) has to be approximated with a rational expression, for example, a Padé approximation. In this case, the stability criterion is only approximative.

The description below assumes that \( a_0 > 0 \) (if \( a_0 < 0 \), we change the sign of all coefficients); often we have \( a_0 = 1 \). The stability of the system is determined as follows:

1. If any coefficient is non-positive (i.e., zero or negative) we can immediately say that the system is unstable. This follows from the fact that the characteristic equation then must have at least one zero (and the system thus at least one pole) with a non-negative real part.

2. If all the coefficients are positive the system can be stable, but no firm conclusion can yet be made.
6.3.1 Routh-Hurwitz’s stability criterion

A **sufficient and necessary stability condition** is obtained by using the following scheme:

\[
\begin{align*}
  a_0 & \quad a_2 & a_4 & \ldots \\
  a_1 & \quad a_3 & a_5 & \ldots \\
  c_0 & \quad c_1 & c_2 & \ldots \\
  d_0 & \quad d_1 & d_2 & \ldots \\
  \vdots & \quad \vdots & \vdots & \ddots
\end{align*}
\]

\[
\begin{align*}
  c_0 &= \frac{a_1 a_2 - a_0 a_3}{a_1}, &
  c_1 &= \frac{a_1 a_4 - a_0 a_5}{a_1}, &
  \quad \ldots, &
  c_i &= \frac{a_1 a_{2i+2} - a_0 a_{2i+3}}{a_1} \\
  d_0 &= \frac{c_0 a_3 - a_1 c_1}{c_0}, &
  d_1 &= \frac{c_0 a_5 - a_1 c_2}{c_0}, &
  \quad \ldots, &
  d_i &= \frac{c_0 a_{2i+3} - a_1 c_{i+1}}{c_0}
\end{align*}
\]  

(6.16)

The coefficients in the Routh-Hurwitz’table to the left in (6.16) are obtained as follows:

- As can be seen, the elements in the **first two rows** in the table are obtained directly from the characteristic equation. If the second row contains a coefficient less than the first row, a zero is introduced as the last element so that both rows have the same number of elements.

- The elements of the **third and fourth row** are obtained according to the formulas to the right in (6.16). If some element needed in a formula is missing from the table (i.e., its position is to right of the last element in a row), it is set equal to zero. (The calculated element will then also be equal to zero.)
6.3.1 Routh-Hurwitz’ stability Criterion

- Elements in the following rows (i.e., 5\textsuperscript{th}, 6\textsuperscript{th}, etc. row) are calculated according to the same principle used to calculate the elements in the third and fourth row — an element in column \( j \) is obtained by crosswise multiplications of the elements in the first column and column \( j + 1 \) of the two previous two rows, while the denominator is equal to the first column element in the previous row. For a system of \( n \textsuperscript{th} \) order, a table with \( n + 1 \) rows is obtained (of which \( n - 1 \) are calculated).

- If the first element in a row is equal to zero when there is another element in the row that may be different from zero, the first element is replaced by \( \varepsilon \) (a small positive number), which is then used in further calculations. Once all the elements in the table are determined, the elements that contain \( \varepsilon \) get the values obtained by letting \( \varepsilon \to 0 \).

\textit{The stability condition is that all the elements in the first column of the table have to be strictly positive.}

If any element in the first column is non-positive, the system is unstable; \textit{the number of sign changes in the first column are equal to the number of the system poles with positive real part.}
6.3.1 Routh-Hurwitz’ stability Criterion

Remark 1. During the calculations, it may sometimes become clear that all remaining elements must be equal to zero. We can then, of course, stop the calculations.

Remark 2. If an element in the first column is equal to zero, this corresponds to a pole with the real part equal to zero.

Remark 3. The stability criterion that all elements in the first column have to be positive, can be used to calculate stability limits with respect to unknown parameter values which are included in the characteristic equation, for example, controller parameters if the system is a feedback system.
6.3.1 Routh-Hurwitz’ stability Criterion

**Exercise 6.3**
Show that the following stability conditions apply when the characteristic equation is in the form of (6.12) with $a_0 = 1$.

(a) An arbitrary second order system is stable if and only if $a_1 > 0$ and $a_2 > 0$.
(b) An arbitrary third order system is stable if and only if $a_1 > 0$, $a_3 > 0$ and $a_1a_2 > a_3$.

**Exercise 6.4**
Investigate if the feedback system to the right is stable, and if it is unstable, how many poles it has in the right-half complex plane.

**Exercise 6.5**
Solve exercise 6.2 by means of Routh-Hurwitz’ s stability criterion.
6.3.1 Routh-Hurwitz’ stability Criterion

**Exercise 6.6**

For which values of the controller gain \( K_c \) is the system below stable?

\[
G_p = \frac{1}{5s+1}, \quad G_v = \frac{1}{2s+1}, \quad G_m = \frac{1}{s+1}, \quad C = K_c
\]

**Exercise 6.7**

Investigate with the R-H criterion for which values of the controller gain \( K_c \) a feedback system with the same structure as the system above is stable when

\[
G_p = \frac{4e^{-2s}}{5s+1}, \quad G_v = 0.5, \quad G_m = 1, \quad C = K_c
\]

Replace the time delay with a Padé approximation of first order.
6.3 Analysis methods

6.3.2 Determination of the stability limit by direct substitution

When the poles of a system are located in the complex plane, the imaginary axis represents the stability limit.

- When a system is at the stability limit, at least one pole of the system is located on the imaginary axis.

- Such a pole, which has the form $s = \pm j\omega$ (where $\omega$ can even be zero), must satisfy the characteristic equation at the instability limit.

- If the characteristic equation contains unknown parameters, for example controller parameters, the values of these parameters at the stability limit can be determined.

- Time delays can be treated exactly, as the analysis below shows.
6.3.2 Determination of the stability limit by direct substitution

Substitution of \( s = \pm j\omega \) into the characteristic equation \( A(s) = 0 \) gives, after application of \( j^2 = -1 \) and rearrangement, an expression of the form

\[
A(j\omega) = C(\omega) + jD(\omega) = 0 ,
\]

(6.17)

where \( C \) and \( D \) are functions of \( \omega \) and possible unknown parameters. The system of equations,

\[
\begin{cases}
C(\omega) = 0 \\
D(\omega) = 0
\end{cases}
\]

(6.18)

then yields \( \omega = \omega_c \) and an expression of the possible unknown parameters that define the stability limit with respect to these parameters.

A time delay \( e^{-Ls} \) causes no problems, since we can use Euler’s formula

\[
e^{-j\omega L} = \cos(L\omega) - jsin(L\omega)
\]

(6.19)
6.3.2 Determination of the stability limit by direct substitution

**Exercise 6.8**
Solve exercise 6.6 by direct substitution of \( s = \pm j\omega \).

**Exercise 6.9**
Solve exercise 6.7 by direct substitution without approximation of the time delay.