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6. Stability

As shown in the first two chapters, a successful controller design requires compromises between performance ("speed") and stability.

- An uncontrolled system that is stable may become unstable by aggressive control.
- On the other hand, there are also systems that are unstable without control which can be stabilized by control.

We can conclude that stability is necessary, but not sufficient, for good control.

It is obvious that systematic methods are needed to determine if a system — controlled or uncontrolled — is stable or unstable.
6. Stability

6.1 Stability definitions

Stability can be defined in many different ways.

- For all practical purposes, the different definitions are equivalent for linear systems.
- In a given situation, a certain definition may be more convenient to use than other definitions.

Therefore, it is useful here to mention the most common stability definitions.

The two following, quite concrete definitions, are general in so far as they

- apply for linear and nonlinear systems;
- are independent of the type of system description (transfer function or state-space model).
6. Stability

6.1 Stability definitions

6.1.1 Asymptotic stability

A system is asymptotically stable if it returns to its initial state after a transient disturbance.

- A typical transient disturbance is a pulse, and in practice, many calculations become easier if the pulse is assumed to be an impulse.
- A step change is not a transient disturbance.

**Remark 1.** Asymptotic stability is often defined in more mathematical terms than the above. Although the definitions might seem different, they are equivalent.

6.1.2 Input-output stability

A system is input-output stable if a limited input signal results in a limited output signal.

- A typical limited input signal is a step change.

**Remark 2.** It follows from the definition that an input-output stable system has a finite gain at all “frequencies” (see Chapter 8).
6. Stability

6.2 Poles and stability

To be useful in the mathematical analysis and design, the stability definitions have to be formulated in more mathematical terms.

We will here consider the time response (the transient response) of an arbitrary system (without time delay) when it is subject to

- (i) a transient
- (ii) a permanent change of the input signal.

6.2.1 The time response of a linear system

According to Section 4.3 and Eq. (4.30), the transfer function of a system without a time delay can generally be written

\[
G(s) = \frac{b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}
\]

(6.1)

where

\[
A(s) \equiv s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n
\]

(6.2)

is the characteristic polynomial of the system.
6.2 Poles and stability

Assume that the characteristic polynomial can be *factorized* as

\[ A(s) = (s - p_1)(s - p_2) \ldots (s - p_n) \]  

(6.3)

where \( p_k, k = 1,2, \ldots, n \), are the *zeros* of the characteristic polynomial, which are also *poles of the system*.

The form of the time response depends on whether the poles are

- real or complex
- distinct or repeated

See section 4.4.2 Partial fraction expansion (PFE).

**Distinct real poles**

If the system is *strictly proper* (i.e., \( m < n \)) there exists a PFE

\[ G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n} \]  

(6.4)

where the constants \( c_k, k = 1,2, \ldots, n \), can be determined as described in Section 4.4.2. The output signal \( Y(s) \) of the system is given by

\[ Y(s) = \left( \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n} \right) U(s) \]  

(6.5)

where \( U(s) \) is the input signal to the system.
6.2.1 The time response of a linear system

The input signal is an impulse

If the inputs signal is an impulse, it is a transient disturbance as in the definition of asymptotic stability.

The Laplace transform of the impulse is \( U(s) = I \). Substitution into (6.5) and application of the inverse Laplace transform give

\[
y(t) = C_1 I e^{p_1 t} + C_2 I e^{p_2 t} + \cdots + C_n I e^{p_n t}, \quad t \geq 0
\]

(6.6)

The condition for asymptotic stability is that \( y(t) \to 0 \) when \( t \to \infty \). This happens if and only if all \( p_k < 0 \), \( k = 1, 2, \ldots, n \).

The input signal is a step change

If the inputs signal is a step change, it is a permanent disturbance as in the definition of input-output stability.

The Laplace transform of a step change is \( U(s) = u_{\text{step}}/s \), where \( u_{\text{step}} \) is the step size. Substitution into (6.5) and the inverse Laplace transform give for \( p_k \neq 0 \)

\[
y(t) = C_1 u_{\text{step}} (1 - e^{p_1 t}) + \cdots + C_n u_{\text{step}} (1 - e^{p_n t}), \quad t \geq 0
\]

(6.7)

The condition for input-output stability is that \( y(t) \) remains limited as \( t \to \infty \). This happens if and only if all \( p_k < 0 \), \( k = 1, 2, \ldots, n \).

If any \( p_k = 0 \), the inverse transform gives another solution, where \( y(t) \) grows with time \( t \). Then the system is not input-output stable.
6.2 Poles and stability

6.2.1 The time response of a linear system

**Distinct complex zeros**

Complex zeros of the characteristic polynomial occur as *complex-conjugated pairs*. In a partial fraction expansion, one can choose between

- combining such pairs into a factor of second order (see section 4.4.2)
- calculating with complex numbers (see below)

Let a complex-conjugated pair of zeros be $p_1 = \sigma + j\omega$ and $p_1 = \sigma - j\omega$. The first two terms on the right-hand side of (6.6) give

$$ y_{1+2}(t) = C_1 e^{(\sigma+j\omega)t} + C_2 e^{(\sigma-j\omega)t} = I e^{\sigma t} (C_1 e^{j\omega t} + C_2 e^{-j\omega t}) $$

$$ = I e^{\sigma t} ((C_1+C_2) \cos \omega t + j(C_1-C_2) \sin \omega t), \ t \geq 0 $$

(6.8)

where the last equality follows from Euler’s formula.

- Because the signal $y_{1+2}(t)$ must be real, it follows that $C_1 + C_2$ must be real and $C_1 - C_2$ must be imaginary. This requirement is satisfied if $C_1$ and $C_2$ are complex conjugates.

- Because the trigonometric functions in (6.8) are limited (finite), $y_{1+2}(t) \to 0$ when $t \to \infty$ if and only if $\sigma < 0$.

- Thus, the system is asymptotically stable if and only if $\text{Re}(p_k) < 0$.

- The same condition can be derived for input-output stability using a step change as input signal.
Repeated zeros

If the characteristic polynomial contains repeated (multiple) zeros, either real or complex-conjugated, the following applies.

- The inverse Laplace transform of the PFE will, besides similar terms as in (6.6), (6.7) and (6.8), contain products of exponential functions and the time $t$ raised to a certain power.

- Because the exponential function $e^{p_k t}$, with $\text{Re}(p_k) < 0$, decreases faster than $t^n$ grows for any power $n$, such terms will converge towards zero when $t \rightarrow \infty$.

From this it follows that the stability conditions derived for distinct poles also apply when the system has repeated poles.
6.2.2 Stability conditions in terms of system poles

According to the analysis above, the stability condition can be expressed by the system poles as follows.

A continuous-time system is *stable if and only if all system poles* $p_k$, $k = 1, 2, \ldots, n$, *are located in the left half of the complex plane*, i.e., if

$$\text{Re}(p_k) < 0, \quad k = 1, 2, \ldots, n$$

(6.9)

The system poles are the zeros of the characteristic equation $A(s) = 0$.

**Remark 3.** For linear systems stability is a *system property*. This means that

- if the stability condition is fulfilled for *any* transient or bounded input signal, the stability condition is fulfilled for *all* such input signals.

For nonlinear systems, this need not be the case.
6.2.3 Feedback systems

Obviously, the stability condition (6.9) also applies to feedback (controlled) systems.

Fig. 6.1 shows a simple feedback system with the following transfer functions:

- $G_p(s)$ system being controlled
- $G_m(s)$ measuring device
- $G_c(s)$ controller

Standard block diagram algebra gives

$$Y = \frac{G_p G_c}{1 + G_p G_c G_m} R + \frac{1}{1 + G_p G_c G_m} V$$

where $Y$ is the output, $R$ is the setpoint, and $V$ is an output disturbance.

Here we have

- the loop transfer function $G_\ell = G_p G_c G_m$ (6.11)
- the characteristic equation $1 + G_\ell = 0$ (6.12)
6.2 Poles and stability

The loop transfer function is of the form

\[ G_\ell(s) = \frac{B_\ell(s)}{A_\ell(s)} e^{-L_\ell s} \]  

(6.13)

where \( A_\ell(s) \) and \( B_\ell(s) \) are polynomials in \( s \) and \( L_\ell \) is a possible time delay.

Substitution of (6.12) into (6.12) gives the characteristic equation

\[ A(s) = A_\ell(s) + B_\ell(s)e^{-L_\ell s} = 0 \]  

(6.14)

- If there is no time delay (i.e., \( L_\ell = 0 \)), (6.14) is a pure polynomial. Stability can then be analysed by Routh-Hurwitz’s method (Section 6.3.2).
- If there is a time delay (i.e., \( L_\ell > 0 \)), Routh-Hurwitz’s method cannot be directly used. The options are then to
  - approximate the time delay by a rational expression (Section 5.4) and use Routh-Hurwitz’s method for an approximate stability analysis
  - use direct substitution (Section 6.3.3)
  - use Bode’s stability criterion (Section 8.3)

For a feedback loop to be stable, all signals in the loop must be bounded. Instability of a signal other than the output may occur if the denominator of \( G_p \), \( G_m \) or \( G_c \) contains an unstable factor \( s - p, p > 0 \), which also appears in the numerator of another of the three transfer functions.

- To detect such instability, \( s - p, p > 0 \), must not be cancelled out in \( G_\ell(s) \).
6. Stability

**Exercise 6.1**
Show that the system $G_p(s) = \frac{10}{s-1}$ is unstable. Determine if it can be stabilized by a P controller.

**Exercise 6.2**
Is the system $G_p(s) = \frac{1}{s^2+2s+2}$ stable or unstable? Determine if the closed-loop system is stable when the system is controlled by a PI controller with the parameters
(a) $K_c = 1, T_i = 0.5$
(b) $K_c = 15, T_i = 0.5$
(c) $K_c = 15, T_i = 0.25$
6. Stability

6.3 Analysis methods

6.3.1 Overview

Use of the stability criterion, defined by the system poles, requires that the poles can be determined.

- For systems of higher order than 2, it can be difficult or even impossible to determine the poles analytically, but if there are no unknown parameters, the poles can be calculated numerically.

- Often it is of interest to investigate the stability limit as a function of one or more unknown parameters (e.g., controller parameters), preferably so that the limit can be expressed by one or more analytical expressions. This is problematic for high-order systems.

- Another complication arises if the system contains a time delay which appears in the characteristic equation. This situation occurs if a feedback-controlled system has a time delay.

For these reasons, various methods for stability analysis have been developed. The following methods are studied in this course.
6.3 Analysis methods

### Routh-Hurwitz’s stability criterion

This method, which is studied in Section 6.3.2, can give stability intervals with respect to unknown parameter values, for example, controller parameters. High system orders cause no special problems, but time delays cannot be handled accurately.

### Stability analysis by “direct substitution”

This method, which is studied in Section 6.3.3, is based on the fact that the system poles, i.e., the zeros of the characteristic equation, must be located on the imaginary axis of the complex plane at the stability limit. Stability intervals with respect to unknown parameter values can be obtained. Time delays can be handled accurately, but the calculations for systems of high order are difficult.

### Bode’s stability criterion

This method, which is studied and applied in Chapter 8, is a so-called frequency-domain method, which can handle time delays without approximation. The analysis can be done “graphically” or numerically.
6. Stability

6.3 Analysis methods

6.3.2 Routh-Hurwitz’s stability criterion

Use of the Routh-Hurwitz stability criterion requires that the characteristic equation can be written as a polynomial,

\[ A(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \]  

(6.15)

As noted, a possible time delay \( e^{-Ls} \) has to be approximated by a rational expression, e.g., a Padé approximation (Section 5.5). In this case, the stability analysis is only approximate.

The description below assumes that \( a_0 > 0 \) (if \( a_0 < 0 \) initially, the sign of all coefficients can be changed); often \( a_0 = 1 \). The stability of the system is determined as follows.

- If any coefficient in (6.15) is non-positive (i.e., zero or negative), the system is unstable. This follows from the fact that the characteristic equation then must have at least one zero (and the system thus at least one pole) with a non-negative real part.

- If all coefficients are positive, the system can be stable, but no firm conclusion can yet be made. The analysis continues as follows.
6.3 Analysis methods

- A **sufficient and necessary stability condition** is obtained by forming the Routh-Hurwitz table. For an \( n \)th order system, the table contains \( n \) rows. If an element needed in a calculation below is missing, the value zero is used.

- The elements in the first two rows of the RH table are obtained directly from the characteristic equation as illustrated. Row 1 contains all coefficients with even numbered subscripts, row 2 all odd-numbered subscripts.

- If \( n \geq 2 \), the elements of row 3 are calculated according to (6.16a) so that the number of elements is one less than in row 1,

\[
c_0 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad c_1 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \ldots, \quad c_i = \frac{a_1 a_{2i+2} - a_0 a_{2i+3}}{a_1}
\]  

(6.16a)

- If \( n \geq 3 \), the elements of row 4 are calculated according to (6.16b) so that the number of elements is one less than in row 2,

\[
d_0 = \frac{c_0 a_3 - a_1 c_1}{c_0}, \quad d_1 = \frac{c_0 a_5 - a_1 c_2}{c_0}, \ldots, \quad d_j = \frac{c_0 a_{2j+3} - a_1 c_{j+1}}{c_0}
\]  

(6.16b)
6.3 Analysis methods

6.3.2 Routh-Hurwitz’s stability criterion

- If \( n \geq 4 \), the elements of row 5 are calculated according to (6.16c) so that the number of elements is one less than in two rows above,

\[
e_0 = \frac{d_0 c_1 - c_0 d_1}{d_0}, \quad e_1 = \frac{d_0 c_2 - c_0 d_2}{d_0}, \ldots, \quad e_k = \frac{d_0 c_{k+1} - c_0 d_{k+1}}{d_0}
\]  

(6.16c)

- If \( n \geq 5 \), the calculations are continued according to the same principle as (6.16c). An element in column \( m \) is obtained as the difference between crosswise multiplications of the elements in the first column and column \( m + 1 \) in the two previous rows divided by the first element in the previous row.

- If the element to be divided with is zero when row elements have to be calculated, it is replaced by \( \varepsilon \) (a small positive number), which is then used in the calculations. Once all the elements in the table are determined, the elements that contain \( \varepsilon \) get the values obtained by letting \( \varepsilon \to 0 \).

The stability condition is that all the elements in the first column of the R-H table have to be strictly positive.

If any element in the first column is non-positive, the system is unstable; the number of sign changes in the first column are equal to the number of the poles with positive real part.
Remark 1. During the calculations, it may sometimes become clear that all remaining elements must be equal to zero. Then, of course, the calculations can be stopped.

Remark 2. If an element in the first column is equal to zero, it corresponds to a pole with the real part equal to zero.

Remark 3. The stability criterion that all elements in the first column have to be positive, can be used to calculate stability limits with respect to unknown parameter values which are included in the characteristic equation. A typical example is controller parameters if the system is a feedback system.

Exercise 6.3
Show that the following stability conditions apply when the characteristic equation is in the form of (6.15) with $a_0 = 1$:

(a) An arbitrary second order system is stable if and only if $a_1 > 0$ and $a_2 > 0$.

(b) An arbitrary third order system is stable if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$. 
6.3 Analysis methods

6.3.2 Routh-Hurwitz’s stability criterion

**Exercise 6.4**
Determine if the feedback system in the figure is stable, and if it is unstable, how many poles it has in the right-half complex plane.

**Exercise 6.5**
Solve Exercise 6.2 by means of Routh-Hurwitz’s stability criterion.

**Exercise 6.6**
For which values of the controller gain $K_c$ is the system in the figure stable when $G_p = \frac{1}{5s+1}, G_v = \frac{1}{2s+1}, G_m = \frac{1}{s+1}, C = K_c$?

**Exercise 6.7**
Investigate with the R-H criterion for which values of the controller gain $K_c$ a feedback system with the same structure as in Exercise 6.6 is stable when $G_p = \frac{4e^{-2s}}{5s+1}, G_v = 0.5, G_m = 1, C = K_c$. Use (a) a first-order, (b) a second-order, Padé approximation for the time delay.
6.3.3 Finding the stability limit by direct substitution

Because the poles of a stable system are located in the left-half complex plane (LHP), \textit{the imaginary axis represents the stability limit.}

- When a system is at the stability limit, at least one pole of the system is located on the imaginary axis.
- Such a pole, which has the form \( s = \pm j\omega \) (where \( \omega \) can be zero), must satisfy the characteristic equation at the instability limit.
- If the characteristic equation contains unknown parameters, for example controller parameters, the values of these parameters can be determined at the stability limit.
- \textit{Time delays can be treated exactly}, as the analysis below shows.

Substitution of \( s = j\omega \) into the characteristic equation \( A(s) = 0 \) gives, after application of \( j^2 = -1 \) and rearrangement, an expression of the form

\[
A(j\omega) = C(\omega) + jD(\omega) = 0
\]  

(6.17)

where \( C \) and \( D \) are functions of \( \omega \) and possibly unknown parameters.
Equation (6.17) requires
\[
\begin{align*}
C(\omega) &= 0 \\
D(\omega) &= 0
\end{align*}
\]  
which can be solved to obtain \( \omega = \omega_c \). If \( C \) and/or \( D \) contain an unknown parameter, the value of this parameter at the stability limit is also obtained.

A time delay \( e^{-Ls} \) causes no problems in principle, because Euler’s formula
\[
e^{-L\omega j} = \cos L\omega - j \sin L\omega
\]  
can be used.

**Exercise 6.8**
Solve Exercise 6.6 by the method of direct substitution.

**Exercise 6.9**
Solve Exercise 6.7 by direct substitution without approximation of the time delay.