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4. Laplace Transform Methods

4.1 Linear systems

In practice, all real processes (systems) are nonlinear to some degree. However, there are several reasons why we want to study linear systems.

- It is often difficult to include the right nonlinearity in a model; a linear model might be the best available approximation.
- A system operating close to a stationary operating point — as controlled systems tend to do — often behaves as a linear system.
- There are powerful methods based on linear algebra and operator theory for analysis, synthesis and design of linear systems.

As we have seen, differential equations

- give a mathematical description of continuous-time dynamical systems
- describe how a given variable, the output, depends on one or several other variables, inputs

Linear DEs are therefore suitable for describing linear continuous-time systems mathematically.
4.1.1 Linear differential equations

A linear ODE has the general form

\[ a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_{m-1} \frac{du}{dt} + b_m u \]  

(4.1)

- \( y \) is the **output** from the system, \( u \) is the **input** to the system.
- \( n \), the order of the highest output derivative, is the **system order**.
- The system is **proper** if \( n \geq m \), it is **strictly proper** if \( n > m \); physical systems are practically always proper (but an ideal controller might be **nonproper**).
- The coefficients \( a_0, a_1, \ldots, a_{n-1}, a_n, b_0, b_1, \ldots, b_{m-1}, b_m \) are **system parameters** that completely characterize the properties of the system.

The system parameters can be rescaled (if desired) by multiplying (or dividing) them all by the same factor. Rescaling to get

- \( a_0 = 1 \) is always possible (if \( a_0 = 0 \), the system is not \( n \)th order)
- \( a_n = 1 \) is possible if the static gain is nonzero (which usually applies)
4.1 Linear systems

If the system is proper, we can always use \( n \) as the order of the highest derivative of the input by letting the coefficients \( b_0, \ldots, b_{n-m-1} \) be zero in

\[
\begin{align*}
a_0 \frac{d^ny}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y \\
= b_0 \frac{d^nu}{dt^n} + \cdots + b_{n-m-1} \frac{d^{m+1}u}{dt^{m+1}} + b_{n-m} \frac{d^mu}{dt^m} + \cdots + b_{n-1} \frac{du}{dt} + b_n u
\end{align*}
\]

Thus, we can without loss of generality write

\[
\begin{align*}
a_0 \frac{d^ny}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y \\
= b_0 \frac{d^nu}{dt^n} + b_1 \frac{d^{n-1}u}{dt^{n-1}} + \cdots + b_{n-1} \frac{du}{dt} + b_n u
\end{align*}
\]

(4.2)

**Note** how the subscripts of the coefficients are related to the order of the corresponding time derivative. (This will be useful later.)
4.1.2 Principle of superposition

For linear systems, the *principle of superposition* applies. Assume that

\[ y_1(t) = F(u_1(t)) \]  \hspace{1cm} (4.3a)
\[ y_2(t) = F(u_2(t)) \]  \hspace{1cm} (4.3b)

are two solutions to (4.2). According to the principle of superposition, the input

\[ u(t) = \alpha u_1(t) + \beta u_2(t) \]  \hspace{1cm} (4.4)

where \( \alpha \) and \( \beta \) are arbitrary constants, gives the solution

\[ y(t) = \alpha y_1(t) + \beta y_2(t) \]  \hspace{1cm} (4.5)

For a linear system with more than one input, this implies that we can consider one input at a time. The combined effect of several inputs is then obtained by combining the respective outputs in the same way.

This is a reason why it is (usually) sufficient to include only one input in the DE (4.1) or (4.2).
4. Laplace Transform Methods

4.2 The Laplace transform

It is possible to solve the linear ODE (4.1) or (4.2) “analytically” using basic mathematics if

- the system parameters are constant
- the input \( u(t) \) has a reasonable simple form

The full solution, i.e. the function \( y(t) \), is obtained as the sum of

- a particular solution (any solution satisfying the DE), and
- the general solution to the corresponding autonomous DE \( (u(t) \equiv 0) \)

However, this way of solving DEs is cumbersome:

- the mathematics tend to be complicated for systems of high order
- there are no convenient short-cuts do deal with systems composed of simple subsystems

The **Laplace transform** offers a practical way of solving linear DEs. Furthermore, it plays a fundamental role in analysis, synthesis, and design of linear systems. It is especially useful for systems with one output (and one input).
4.2.1 Definition

The signals in dynamical systems are functions of time. Consider a fairly arbitrary function \( f(t) \). For “technical reasons” concerning the Laplace transform we need to assume that

- \( f(t) = 0 \) for \( t < 0 \), \( f(t) \) can be integrated for \( t \geq 0 \).

The Laplace transform \( F(s) = \mathcal{L}\{f(t)\} \) of such a time function \( f(t) \) is defined by the integral

\[
F(s) = \mathcal{L}\{f(t)\} \equiv \int_0^\infty e^{-st} f(t) dt
\]

where \( s \) is a complex variable, whose real part \( \text{Re}(s) \) has to be large enough for the integral to have a finite value.

- \( f(t) \) is a function in the time domain.
- \( F(s) \) is a function in the Laplace domain or \( s \) domain.
- It is recommended to use small (lower-case) letters for functions in the time domain and the corresponding large (upper-case) letter for a function in the Laplace domain.
  - This recommendation is not always obeyed; we might use the same letter followed by the domain variable \( t \) or \( s \) as argument (e.g. \( F(t) \) and \( F(s) \)).
4.2 The Laplace transform

**The inverse Laplace transform**

In order for the Laplace transform to be useful, it is necessary that we can also transform the other way, i.e. to calculate the time function $f(t)$ that corresponds to a given Laplace function $F(s)$.

Formally, this can be done by means of the integral formula

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \equiv \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} F(s) ds, \quad t \geq 0$$

(4.7)

where $j = \sqrt{-1}$ is the imaginary unit and $\sigma$ is a real number large enough so that $F(s)$ is finite for all $\text{Re}(s) > \sigma$.

In practical calculations

- (4.7) is not needed
- (4.6) is seldom needed

because *tabulated Laplace transform/time domain function pairs can be used* (see Laplace Transform Table, LTT, Section 4.5). However, there is a need to apply

- the superposition principle
- partial fraction expansion (section 4.4.2)

to be able to use tabulated function relationships.
4.2.2 Laplace transforms of common functions

A rectangular pulse

A rectangular pulse has a
- constant amplitude (height) \( a \)
- given pulse length \( T \)

It is assumed to start at \( t = 0 \).

By means of the Laplace transform definition (4.6) we can derive

\[
F(s) = \int_0^T e^{-st} a \, dt = a \left[ -\frac{1}{s} e^{-st} \right]_0^T = a \frac{1-e^{-sT}}{s} \tag{4.8}
\]

A \textit{unit pulse} has the area 1, i.e. \( aT = 1 \). The Laplace transform of the unit pulse is thus

\[
F(s) = \frac{1-e^{-sT}}{sT} \tag{4.9}
\]
4.2 The Laplace transform

The unit impulse — Dirac’s delta function

An impulse is defined as a (rectangular) pulse, whose

- pulse length \( T \to 0 \)
- amplitude \( a \to \infty \)
- \( aT \) is finite

For the unit impulse, denoted \( \delta(t) \),

- \( aT = 1 \) (with some physical unit according to the application)

The Laplace transform of the unit impulse can be obtained by letting \( T \to 0 \) in the Laplace transform of the unit pulse. A Taylor series expansion then gives

\[
\mathcal{L}\{\delta(t)\} = \lim_{T \to 0} \frac{1 - e^{-sT}}{sT} = \lim_{T \to 0} \frac{sT - \frac{1}{2}(sT)^2 + \cdots}{sT} = \lim_{T \to 0} \left[ 1 - \frac{1}{2} (sT) \right] = 1
\]

\[
\mathcal{L}\{\delta(t)\} = 1 \tag{4.10}
\]

- Impulses are important in many practical applications. Input signals of very short duration can usually be approximated by impulses. E.g.
  - voltage and current pulses in electrical systems
  - sudden forces in mechanical systems, e.g. wind gusts
  - injection of tracers in medical and process technical applications
4.2 The Laplace transform

4.2.2 Laplace transforms of common functions

**A unit step**

A *step function* can be considered a rectangular pulse with
- infinite pulse length $T$

For a *unit step*, denoted $\sigma(t)$,
- $a = 1$ (with some physical unit)

The Laplace transform of a unit step can be derived by letting $T \to \infty$ in the Laplace transform of a pulse with $a = 1$. A Taylor series expansion gives

$$\mathcal{L}\{\sigma(t)\} = \lim_{T \to \infty} \frac{1 - e^{-sT}}{s} = \frac{1}{s} \quad (4.11)$$

**A unit ramp**

A *ramp function* is a function whose value changes linearly with time. For a *unit ramp*, denoted $\rho(t)$, the
- slope coefficient is 1, i.e. $\rho(t) = t$, $t \geq 0$.

The Laplace transform of the unit ramp can be derived from the definition (4.6). Integration by parts gives

$$\mathcal{L}\{\rho(t)\} = \frac{1}{s^2} \quad (4.12)$$
4.2 The Laplace transform

Relationships between the simple unit functions

Consider the unit impulse, unit step, and unit ramp in Fig. 4.2.

- the impulse is the time derivative of the step function
- the step function is the time derivative of the ramp function

Mathematically:

\[ \delta(t) = \frac{d}{dt} \sigma(t), \quad \sigma(t) = \frac{d}{dt} \rho(t) \]

In the Laplace domain we have derived the relationships:

\[ \mathcal{L}\{\delta(t)\} = 1 = s \frac{1}{s} = s \mathcal{L}\{\sigma(t)\}, \quad \mathcal{L}\{\sigma(t)\} = \frac{1}{s} = s \frac{1}{s^2} = s \mathcal{L}\{\rho(t)\} \]

This means that

- a time derivative corresponds to multiplication by \( s \) in the \( \mathcal{L} \)-domain
- thus, a time integral corresponds to division by \( s \) in the \( \mathcal{L} \)-domain

Fig. 4.2. A unit impulse, unit step, and unit ramp.
### Exponential function

An exponential function is defined \( f(t) = e^{-at}, \ t \geq 0 \). If

- \( a > 0 \), the function is **exponentially decaying**
- \( a < 0 \), the function is **exponentially increasing**

The Laplace transform can be derived from the definition (4.6)

\[
F(s) = \mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} \, dt \\
= \int_0^\infty e^{-(s+a)t} \, dt = \left[-\frac{1}{s+a} e^{-(s+a)t}\right]_0^\infty \\
\Rightarrow F(s) = \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}
\]

It is not obvious from the final result, but the integral exists (is finite) if and only if

- \( \text{Re}(s) + a > 0 \)

This is a limiting restriction mainly if \( a < 0 \), i.e., if the function is exponentially increasing. Such a function is **unstable**, because it approaches infinity as time goes towards infinity.
4.2 The Laplace transform

Sine and cosine functions

The *sine* function (with amplitude one) is defined $f(t) = \sin(\omega t), \ t \geq 0$, where $\omega$ is the *oscillation frequency* in radians per time unit.

The Laplace transform can be derived using *integration by parts*, defined

$$
\int_{t_1}^{t_2} g'(t)f(t)dt = [g(t)f(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} g(t)f'(t)dt
$$

By applying partial integration twice to the definition (4.6), we can derive

$$
\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2} \quad (4.14)
$$

The *cosine* function (amplitude one) is defined $f(t) = \cos(\omega t), \ t \geq 0$. Similarly as above, we can derive

$$
\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2} \quad (4.15)
$$

We could have derived (4.15) more easily by observing that

$$
\cos(\omega t) = \omega^{-1} \frac{d}{dt} \sin(\omega t)
$$

and using

$$
\mathcal{L}\{\cos(\omega t)\} = \omega^{-1}s\mathcal{L}\{\sin(\omega t)\}
$$
4.2.3 Properties of the Laplace transform

Superposition

If \( \mathcal{L}\{f_1(t)\} = F_1(s) \) and \( \mathcal{L}\{f_2(t)\} = F_2(s) \),

Then

\[
\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s) \quad (4.16)
\]

\[
\alpha f_1(t) + \beta f_2(t) = \mathcal{L}^{-1}\{\alpha F_1(s) + \beta F_2(s)\} \quad (4.17)
\]

where \( \alpha \) and \( \beta \) are arbitrary constants.

This means that we can easily calculate the

- Laplace transform of the linear combination \( f(t) = \alpha f_1(t) + \beta f_2(t) \) according to (4.16)
- inverse Laplace transform of \( F(s) = \alpha F_1(s) + \beta F_2(s) \) according to (4.17)

if we know the Laplace transforms \( F_1(s) \) and \( F_2(s) \) of (the simpler functions) \( f_1(t) \) and \( f_2(t) \), respectively.
4.2 The Laplace transform

4.2.3 Properties of the Laplace transform

**Derivatives**

If \( \mathcal{L}\{f(t)\} = F(s) \), then the Laplace transform of the first-order time derivative \( \dot{f} \equiv \frac{df}{dt} \) is given by

\[
\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0^-)
\]  

(4.18)

where

\[
f(0^-) = \lim_{t \to 0^-} f(t)
\]

i.e. \( f(0^-) \) is the *initial value* of the time domain function \( f(t) \) when \( t \) approaches 0 from the negative side. Negative or positive side makes a difference if \( f(t) \) has a discontinuity at \( t = 0 \) (e.g. a step function).

The Laplace transform of an \( n:th \) order derivative \( f^{(n)} \equiv \frac{d^n f}{dt^n} \) is

\[
\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \dot{f}(0^-) - \cdots - s f^{(n-2)}(0^-) - f^{(n-1)}(0^-)
\]

(4.19)

where \( f(0^-), \dot{f}(0^-), \ldots \) are the values of \( f(t), \dot{f}(t), \ldots \) when \( t \to 0^- \).

**Note** that all initial values are 0 for systems, where the variables are deviations from a steady-state operating point.
4.2 The Laplace transform

**Integrals**

If \( \mathcal{L}\{f(t)\} = F(s) \), the Laplace transform of the time integral of \( f(t) \) is given by

\[
\mathcal{L}\left\{ \int_{0^-}^{t} f(\tau)d\tau \right\} = \frac{1}{s} F(s)
\]  
(4.20)

For a multiple integral, consisting of \( n \) integrals, the Laplace transform is

\[
\mathcal{L}\left\{ \int_{0^-}^{t} \int_{0^-}^{t} \cdots \int_{0^-}^{t} f(\tau)d\tau^n \right\} = \frac{1}{s^n} F(s)
\]  
(4.21)

**Note** that initial values are not an issue in this case.

**Damping**

If \( \mathcal{L}\{f(t)\} = F(s) \), the Laplace transform of an exponentially damped function \( e^{-at}f(t) \) is given by

\[
\mathcal{L}\{e^{-at}f(t)\} = F(s + a)
\]  
(4.22)

**Note** that it is required that \( \text{Re}(s) + a > 0 \), which is a limiting restriction if \( a < 0 \). In practice, \( a > 0 \) (otherwise there is no damping).
### Time delay

If $\mathcal{L}\{f(t)\} = F(s)$, the Laplace transform of the function $f(t - L)$, i.e. the function $f(t)$ delayed by $L$ time units, is given by

$$\mathcal{L}\{f(t - L)\} = e^{-Ls}F(s)$$  \hspace{1cm} (4.23)

![Fig. 4.3. Undelayed and delayed time function.](image-url)
Limit theorems

In general

- small values of the time variable \( t \) correspond to large values of the Laplace variable \( s \)
- and vice versa

However, there is no exact relationship between time and the Laplace variable, except for two limit conditions. They are useful for calculating the initial and final steady-state values of a time function whose Laplace transform is known without doing an inverse Laplace transformation.

**The initial value theorem**

If \( \mathcal{L}\{f(t)\} = F(s) \), where \( F(s) \) is strictly proper,

\[
\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s) \quad (4.24)
\]

Here \( t \to 0^+ \) means that \( t \) approaches zero from the positive side of \( t \).

**The final value theorem**

If \( \mathcal{L}\{f(t)\} = F(s) \), where \( s F(s) \) is bounded for all \( \text{Re}(s) \geq 0 \),

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s) \quad (4.25)
\]
4.2 The Laplace transform

**Exercise 4.1**
Determine the Laplace transform of the function

\[ f(t) = 6 + 8e^{-t} - 5e^{-2t} \]

Validate the result by means of the initial and final value theorems.

**Exercise 4.2**
Determine the time function whose Laplace transform is

\[ F(s) = \frac{2.4e^{-0.8s}}{2s+3.6} \]

Validate the result by means of the initial and final value theorems.

**Exercise 4.3**
Determine the Laplace transform of the delayed saw-tooth pulse in the figure.

![Delayed saw-tooth pulse graph](image)
4.3 Modelling in the Laplace domain

4.3.1 The transfer function

Assume that the DE (4.1) is satisfied by the variable values

\[
(\tilde{y}(n), \tilde{y}(n-1), ..., \hat{y}, \bar{y}, \bar{u}(m), \bar{u}(m-1), ..., \hat{u}, \bar{u})
\]
i.e., this is a solution to the DE.

- If this solution is of particular interest, we can call it a reference state or an operating point.
- This point is often a steady state or an equilibrium point, where all time derivatives are zero.

Any solution \((y(n), y(n-1), ..., \dot{y}, y, u(m), u(m-1), ..., \dot{u}, u)\) to the DE can be related to the reference state by

\[
\begin{align*}
y(n) &= \tilde{y}(n) + \Delta y(n), \quad \ldots, \quad \dot{y} = \dot{\tilde{y}} + \Delta \dot{y}, \quad y = \bar{y} + \Delta y \quad \text{(4.26a)} \\
u(m) &= \tilde{u}(m) + \Delta u(m), \quad \ldots, \quad \dot{u} = \dot{\tilde{u}} + \Delta \dot{u}, \quad u = \bar{u} + \Delta u \quad \text{(4.26b)}
\end{align*}
\]

where the \(\Delta\)-variables indicate deviations from the reference state.
Substitution of (4.26) into the DE (4.1), cancelling out of the reference state (which satisfies the DE separately), and the choice \( a_0 = 1 \), yield

\[
\frac{d^n \Delta y}{dt^n} + a_1 \frac{d^{n-1} \Delta y}{dt^{n-1}} + \cdots + a_{n-1} \frac{d \Delta y}{dt} + a_n \Delta y
\]

\[
= b_0 \frac{d^m \Delta u}{dt^m} + b_1 \frac{d^{m-1} \Delta u}{dt^{m-1}} + \cdots + b_{m-1} \frac{d \Delta u}{dt} + b_m \Delta u
\]

(4.27)

Considering that all initial values of the \( \Delta \)-variables are zero, the Laplace transform of (4.27) gives,

\[
s^n \Delta Y(s) + a_1 s^{n-1} \Delta Y(s) + \cdots + a_{n-1} s \Delta Y(s) + a_n \Delta Y(s)
\]

\[
= b_0 s^m \Delta U(s) + b_1 s^{m-1} \Delta U(s) + \cdots + b_{m-1} s \Delta U(s) + b_m \Delta U(s)
\]

(4.28)

where \( \Delta Y(s) \) and \( \Delta U(s) \) are the Laplace transforms of \( \Delta y(t) \) and \( \Delta u(t) \), respectively.

Thus, the Laplace transform of an \( n^{th} \) order derivative yields a factor \( s^n \) when the initial state is zero.
4.3 Modelling in the Laplace domain

Equation (4.28) can also be written

\[(s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n)\Delta Y(s)\]

\[= (b_0s^m + b_1s^{m-1} + \cdots + b_{m-1}s + b_m)\Delta U(s)\]

or more compactly

\[\Delta Y(s) = G(s)\Delta U(s)\]  \hspace{1cm} (4.29)

where

\[G(s) = \frac{b_0s^m + b_1s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n} \equiv \frac{B(s)}{A(s)}\]  \hspace{1cm} (4.30)

is the transfer function of the system described by the DE (4.27).

As (4.29) shows, the output defined in the Laplace domain is obtained when the input, also defined in the Laplace domain, is multiplied by the transfer function of the system in question.

Thus, when working in the Laplace domain, the solution of a DE (i.e., finding the output for a given input) can be obtained by algebraic operations only.
4.3 Modelling in the Laplace domain

In (4.30), \( A(s) \) is the denominator polynomial and \( B(s) \) is the numerator polynomial of the transfer function.

- \( A(s) = 0 \) is the **characteristic equation** of the system.
- The roots (i.e., solutions) of \( A(s) = 0 \) are called system **poles**.
- The roots of \( B(s) = 0 \) are called system **zeros**.

The importance of poles and zeros is treated in Ch. 5 and 6.

If the system contains a **pure time delay**, also called a **dead time**, it takes some time before an input begins to affect an output. If the input signal is \( v \), and there is a time delay of \( L \) time units, \( v(t - L) \) takes the place of \( u(t) \) in (4.27). By using the substitution \( u(t) = v(t - L) \) in (4.27), (4.29) and (4.30) are obtained.

The use of \( \Delta \)-variables and the Laplace transform gives

\[
\Delta U(s) = e^{-Ls} \Delta V(s)
\]  

(4.31)

where \( \Delta V(s) \) is the Laplace transform of \( \Delta v(t) \). The transfer function (TF) from \( \Delta V(s) \) to \( \Delta Y(s) \) is then \( G(s)e^{-Ls} \).

To calculate the TF of a system with a time delay, we can first calculate \( G(s) \) assuming no time delay, then include the time delay as \( G(s)e^{-Ls} \).
4.3 Modelling in the Laplace domain

4.3.1 The transfer function

Example 4.1. Transfer function of a first-order system.

A mercury thermometer can approximately be described by the DE

\[ T \frac{d\vartheta_2}{dt} + \vartheta_2 = \vartheta_1 \]  

(1)

where \( \vartheta_1 \) is the temperature outside the thermometer and \( \vartheta_2 \) is the temperature of mercury inside the thermometer. The time constant \( T \) of the thermometer is \( \approx 2 \) min.

We introduce \( \Delta \)-variables to denote deviations from an equilibrium point \( \vartheta_1 = \bar{\vartheta}_1 \) and \( \vartheta_2 = \bar{\vartheta}_2 \), i.e.,

\[ \vartheta_1 = \bar{\vartheta}_1 + \Delta\vartheta_1 , \vartheta_2 = \bar{\vartheta}_2 + \Delta\vartheta_2 \]  

(2)

where \( \Delta\vartheta_1 \) and \( \Delta\vartheta_2 \) denote the size of the deviations. Substitution into (1) yields

\[ T \frac{d(\bar{\vartheta}_2 + \Delta\vartheta_2)}{dt} + \bar{\vartheta}_2 + \Delta\vartheta_2 = \bar{\vartheta}_1 + \Delta\vartheta_1 \]  

(3)

In the equilibrium point it must apply that \( T \frac{d\bar{\vartheta}_2}{dt} + \bar{\vartheta}_2 = \bar{\vartheta}_1 \) (even more strongly, \( \bar{\vartheta}_2 = \bar{\vartheta}_1 \) and \( d\bar{\vartheta}_2/dt = 0 \)). Combination with (3) gives

\[ T \frac{d\Delta\vartheta_2(t)}{dt} + \Delta\vartheta_2(t) = \Delta\vartheta_1(t) \]  

(4)

where the time argument \( t \) is included for clarity.
4.3.1 The transfer function

The Laplace transform of (4) yields

\[ T(s\Delta\Theta_2(s) - \Delta\theta_2(0^-)) + \Delta\Theta_2(s) = \Delta\Theta_1(s) \]  

(5)

where \( \Delta\Theta_1(s) \) and \( \Delta\Theta_2(s) \) are the Laplace transforms of \( \Delta\theta_1(t) \) and \( \Delta\theta_2(t) \), respectively.

The initial value \( \Delta\theta_2(0^-) \) is the value of \( \Delta\theta_2 \) when the \( t \) approaches zero from the negative side. Since we use \( \Delta \)-variables that indicate deviations from the initial state, \( \Delta\theta_2(0^-) = 0 \). Equation (5) reduces to

\[ Ts\Delta\Theta_2(s) + \Delta\Theta_2(s) = \Delta\Theta_1(s) \]  

(6)

or

\[ \Delta\Theta_2(s) = G(s)\Delta\Theta_1(s) \]  

(7)

where

\[ G(s) = \frac{1}{Ts+1} \]  

(8)

is the transfer function of the mercury thermometer.
Exercise 4.4

A system is described by the differential equation

\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = u \]

where \( u \) and \( y \) indicate deviations from an equilibrium point. Determine the transfer function of the system.
4.3.2 Combination of systems

Block diagrams, and especially combinations of signals in block diagrams, were introduced in Section 2.2. Here some typical block connections are considered.

**Series connection**

*Series* (or *cascade*) connection is a common type of block interconnection. According to (4.29), the connection yields

\[
Y(s) = G_2(s)X(s) = G_2(s)G_1(s)U(s)
\]

from which it follows that

\[
G(s) = G_2(s)G_1(s)
\]  

is the transfer function for a series connection of two systems. Obviously, this can be extended to more than two systems in series.

*Note* that this requires that \( G_1(s) \) and \( G_2(s) \) are not changed by the interconnection, i.e., that the systems are “non-interacting”.

![Series connection diagram](image)

*Fig. 4.4. Series connection.*
Example 4.2. Non-interacting tanks.

The liquid tank in the figure can be described by the model (see Example 3.5)

\[
A \frac{dh(t)}{dt} = F_0(t) - F_1(t), \quad F_1(t) = \beta \sqrt{h(t)}
\]

where \( F_1 \) is a free outflow governed by gravity and the liquid level \( h \). \( \beta \) and \( A \) are constants. Linearization at \( h = \bar{h} \) and replacement of \( h \) by \( F_1 \) as dependent variable yields the transfer function

\[
\frac{\Delta F_1(s)}{\Delta F_0(s)} = G(s) = \frac{1}{Ts+1}, \quad T = \frac{2A\sqrt{\bar{h}}}{\beta}.
\]

The two tanks connected in series have the transfer functions

\[
G_1(s) = \frac{1}{T_1s+1} \quad \text{and} \quad G_2(s) = \frac{1}{T_2s+1}
\]

with \( T_1 = (2/\beta)A_1\sqrt{\bar{h}_1} \) and \( T_2 = (2/\beta)A_2\sqrt{\bar{h}_2} \).

The transfer function from \( F_0 \) to \( F_2 \) is given by (4.32) as

\[
\frac{\Delta F_2(s)}{\Delta F_0(s)} = G(s) = G_2(s)G_1(s) = \frac{1}{(T_1s+1)(T_2s+1)}.
\]
4.3.2 Combination of systems

**Interacting systems**

In the modelling of series-connected systems, it is important to know if they are interacting or non-interacting. A series connection is

- **non-interacting**, if each subsystem is only affected by previous subsystems in the series
- **interacting**, if a subsystem is affected by a subsequent subsystem in the series

The transfer function from the input to the first subsystem to the output from the last subsystem in a series connection is

- the product of the transfer functions of the individual subsystems if the series connection is non-interacting, Eq. (4.32);
- **not** directly obtained from the transfer functions of the individual subsystems if the series connection is interacting; the complete series connection has to be treated as a whole.
Example 4.3. Interacting tanks.

The liquid-tank series in the figure is interacting because \( F_1 \) (the outflow from tank 1) is affected not only by \( h_1 \) but also by \( h_2 \) (a variable in tank 2). Tank 2 is described by the model

\[
A_2 \frac{dh_2(t)}{dt} = F_1(t) - F_2(t), \quad F_2(t) = \beta \sqrt{h_2(t)}
\]

yielding the transfer function

\[
\frac{\Delta F_2(s)}{\Delta F_1(s)} = G_2(s) = \frac{1}{T_2 s + 1}, \quad T_2 = \frac{2A_2 \sqrt{h_2}}{\beta}
\]

as for non-interacting tanks. However, because \( F_1(t) = \beta \sqrt{h_1(t)} - h_2(t) \) (assuming \( h_1 \geq h_2 \)), \( G_1(s) \) is different from the non-interacting case.

Linearization and elimination \( h_1 \) and \( h_2 \) from the model for tank 1 yields

\[
T_1 \frac{d\Delta F_1(t)}{dt} + \Delta F_1(t) = K_1 \Delta F_0(t) \quad \Rightarrow \quad \frac{\Delta F_1(s)}{\Delta F_0(s)} = G_1(s) = \frac{K_1}{T_1 s + 1}
\]

\[
K_1 = \frac{2A_2 \sqrt{h_2} + \beta}{2(A_1 + A_2) \sqrt{h_2} + \beta}, \quad T_1 = \frac{2A_1 \sqrt{h_1 - h_2}}{\beta} K_1.
\]

Now the transfer function from \( F_0 \) to \( F_1 \) is given by \( G(s) = G_2(s)G_1(s) \).
4.3 Modelling in the Laplace domain

**Parallel connection**

A *parallel connection* is illustrated in Fig. 4.5. It contains a signal branch and a summation. Standard block algebra yields

\[ Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) \]

\[ = (G_1(s) + G_2(s))U(s) \]

from which it follows that

\[ G(s) = G_1(s) + G_2(s) \] (4.33)

is the transfer function for a parallel connection of two subsystems.

An inverse-response system (Section 5.5) is often obtained by a parallel connection of two systems.

![Fig 4.5. Parallel connection.](image-url)
Feedback connection

The most fundamental system interconnection in control theory is (negative) feedback, which is illustrated in Fig. 4.6. Standard block algebra yields

\[ Y(s) = G(s)E(s) = G(s)(R(s) - H(s)Y(s)) \]

from which

\[ Y(s) = \frac{G(s)}{1+G(s)H(s)} R(s) \]

Thus,

\[ G_c(s) = \frac{G(s)}{1+G(s)H(s)} \]  \hspace{1cm} (4.34)

is the transfer function of a feedback connection, where \( G(s) \) is the transfer function in the “forward” direction and \( H(s) \) in the feedback part.

- The product \( G_\ell(s) = G(s)H(s) \) is called the loop-transfer function.
- The equation \( 1 + G_\ell(s) = 0 \) is the characteristic equation of the system.

Fig 4.6. Feedback connection.
Exercise 4.5

Determine the transfer function from the input \( u \) to the output \( y \) in the block diagram below.
4.3.3 Notational conventions

As previously noted for calculations in the Laplace domain, the output of a system is obtained by multiplying the input by the its transfer function — no other terms can be included in the expression (if there is one input).

- By applying the Laplace transform to a DE, such a linear expression is obtained only if the initial values of the signal and derivatives are zero.
- The condition is satisfied automatically when $\Delta$-variables are used because they denote deviations from a reference state that apply at $t = 0^-$, i.e., when time zero is approached from the negative side. (This is relevant if the function is discontinuous at $t = 0$.)
- Since it is an essential requirement in calculations with transfer functions that the signals have the above characteristics, it is considered to be the case even if it is not mentioned. Thus, as shown in Exercise 4.4, the symbol $\Delta$ can be omitted to simplify the notation.
- If $\Delta$-variables with the symbol $\Delta$ are used, it is often to emphasize the physical meaning of the signals. In such cases the symbol without $\Delta$ is often used to denote a “true” physical value of a variable, for example, a measurement in the process.
4.3 Modelling in the Laplace domain

- It is recommended to denote time functions with “small” letters (lower-case letters) and their Laplace transforms with the corresponding “large” letter (capital letter).

- However, it is not unusual to use the same symbol for both the time function and its Laplace transform; sometimes this is due to lack of available letters.
  - This can be done because it is usually clear from the context which the function type is. For example, in calculations with transfer functions, it is clear that the Laplace transform of the signals are used.
  - If there is a danger of misunderstanding, the argument “t” or “s” can be included to indicate the type of function. This may be needed if the same symbol is used for the time function and its Laplace transform.

- It should be noted that time-domain signals and their Laplace transform generally have a physical unit.
  - Mathematical operations should therefore satisfy unit dimensions both in the time domain and the Laplace domain.
  - In particular it should be noted that the gain of a system is not dimensionless if the input and the output have different units.
4. Laplace Transform Methods

4.4 Applying the inverse Laplace transform

The *inverse Laplace transform* is used when we want to find the time-domain function that corresponds to a given Laplace-domain function.

The general procedure for solving time-domain problems in the Laplace domain is illustrated in the figure.

![Flowchart of solving time-domain problems in the Laplace domain](image)

**Fig. 4.7.** Solving time-domain problems in the Laplace domain.
4. Laplace Transform Methods

4.4 Applying the inverse Laplace transform

4.4.1 Solving linear differential functions

A convenient way to solve linear ordinary differential equations is to use Laplace transform methods.

- When the Laplace transform is applied to a differential equation term by term, the Laplace transform of the dependent variable, i.e. the output signal, can be solved by purely algebraic methods.
- If the differential equation describes a dynamic system, it has an input that is also transformed.
- The time function of the dependent variable can then be obtained by using the inverse transform of its Laplace transform.

Tables of Laplace transforms and their corresponding time functions can be used to find Laplace transforms as well as inverse Laplace transforms.

- If the table does not include a needed Laplace function, it can usually, by using a partial fraction expansion, be written as a sum of simpler functions whose time functions are found in the table.
- According to the superposition theorem (see Section 4.2.3), the full time function is then obtained as the sum of the time functions of the simpler Laplace functions.
Example 4.4. An initial-value problem.

Solve the differential equation $\ddot{y} + 5\dot{y} + 6y = 1$ with the initial values $y(0^-) = 0$ and $\dot{y}(0^-) = 1$.

The Laplace transform of the DE with the given initial values is obtained by (4.18) and (4.19) as

$$ (s^2Y(s) - sy(0^-) - \dot{y}(0^-)) + 5(sY(s) - y(0^-)) + 6Y(s) = \frac{1}{s} \quad (1) $$

Substitution of the initial values yields after some rearrangement

$$ Y(s) = \frac{s+1}{s(s^2+5s+6)} = \frac{s+1}{s(s+2)(s+3)} = \frac{1}{(s+2)(s+3)} + \frac{1}{s(s+2)(s+3)} \quad (2) $$

The Laplace transforms having $s + 1$ in the numerator are not listed in the Table in Section 4.5. However, the Laplace transforms of the last two terms are listed in pt 17 and 18. Because of the superposition principle, the inverse transforms of the two terms can be added to obtain

$$ y(t) = \left[\frac{1}{3-2} (e^{-2t} - e^{-3t})\right] + \left[\frac{1}{2\cdot3} + \frac{1}{2(2-3)} e^{-2t} - \frac{1}{3(2-3)} e^{-3t}\right] $$

$$ = \frac{1}{2} e^{-2t} - \frac{2}{3} e^{-3t} + \frac{1}{6} \quad (3) $$

The solution should be verified by checking if it satisfies the DE + initial conditions.
**Example 4.5. Step response of a first-order system.**

A linear first-order system is described by the differential equation

$$ T \frac{dy}{dt} + y = Ku \quad (1) $$

where $u$ is the input, $y$ is the output, $K$ is the *static gain* and $T$ is the *time constant* of the system.

If $u = 0$, $y = 0$ is a solution to the DE. This equilibrium point can be assumed to apply at $t = 0^-$. Laplace transformation of (1) then gives

$$ Y(s) = G(s)U(s) \quad (2) $$

where

$$ G(s) = \frac{K}{Ts+1} \quad (3) $$

is the *transfer function* of the system.

Assume that the input $u$ suddenly changes from 0 to $u_{\text{step}}$ at $t = 0$, i.e.,

$$ u(t) = 0, \ t < 0; \ u(t) = u_{\text{step}}, \ t \geq 0 \quad (4) $$

The size of this *step change* is $u_{\text{step}}$ times the size of a unit step. According to pt 1 in the Laplace table (and the superposition principle),

$$ U(s) = \frac{u_{\text{step}}}{s} \quad (5) $$
4.4.1 Solving linear DEs

Substitution of (5) and (3) into (2) gives

\[ Y(s) = \frac{Ku_{\text{step}}}{s(Ts+1)} \]  \hspace{1cm} (6)

Pt 26 in the Laplace transform table now yields the time function

\[ y(t) = Ku_{\text{step}} \left( 1 - e^{-t/T} \right) \]  \hspace{1cm} (7)

which is the *step response* of a first-order system.

**Exercise 4.6**

Determine the unit step response for the system in Exercise 4.4, i.e., the differential equation

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = u \]

Note the similarities and differences with Example 4.4.
4.4.2 Partial fraction expansion

The Laplace transform of a time function \( f(t) \), such as the solution to a differential equation with a given input and given initial conditions, can typically be written in the form

\[
F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \equiv \frac{B(s)}{A(s)}
\]  

(4.35)

For a Laplace function \( Y(s) \) containing a time delay \( L \), so that

\[
Y(s) = F(s)e^{-Ls}
\]  

(4.36)

one can first determine \( f(t) \) from \( F(s) \), then obtain \( y(t) = f(t - L) \) according to the time-delay theorem (4.23).

The time function \( f(t) \) corresponding to the Laplace transform \( F(s) \) can often be found directly in a table (Section 4.5) or after a simple separation of the numerator terms according to the superposition theorem as in Example 4.4.

If this does not solve the problem, \( F(s) \) can be rewritten as a sum of simpler terms by a partial fraction expansion (PFE).
4.4 Applying the inverse Laplace transform

**Extracting a non-strictly proper part from** \( F(s) \)

Most physical systems are proper, but in the rare case when \( F(s) \) is not strictly proper, i.e., \( m \geq n \) in (4.35), the non-strictly proper part has to be extracted before proceeding with the PFE.

This is done by performing a *polynomial long division* of \( F(s) \) to obtain

\[
F(s) = F_0(s) + \frac{B_0(s)}{A(s)}
\]

(4.37)

where \( F_0(s) \) is a polynomial of order \( m - n \geq 0 \), \( A(s) \) is the \( n \)th order denominator polynomial of \( F(s) \), and \( B_0(s) \) is a polynomial of order \( m_0 < n \). Thus, \( B_0(s)/A(s) \) is strictly proper.

According to the superposition principle, the inverse Laplace transform of \( F(s) \) is the sum of the inverse transforms of \( F_0(s) \) and \( B_0(s)/A(s) \).

- The inverse transform \( f_0(t) \) of \( F_0(s) \) will consist of one or more terms such as an impulse and time derivatives of impulses (a factor \( s^i \) results in an \( i \)th order derivative).
- The inverse transform of \( B_0(s)/A(s) \) can be obtained via a PFE.
Consider the rational function \( F(s) = \frac{s^3 - 2s^2 - 4}{s - 3} \). Write it in the form of (4.37) by doing a polynomial long division. Find the inverse Laplace of \( F(s) \).

The long division procedure yields

\[
B(s) = \frac{s^3 - 2s^2}{s - 3} = \frac{-s^3 + 3s^2}{s^2} + \frac{s^2 + 3s}{s^2} + 4 \frac{3s - 4}{s^2 + s + 3} = A(s) = F_0(s) = B_0(s)
\]

Thus, \( F(s) = s^2 + s + 3 + \frac{5}{s-3} \).

Equations (4.18), (4.19) and pts 6 and 8 in the Laplace Transform Table yield the inverse Laplace transform \( f(t) = \frac{d^2 \delta(t)}{dt^2} + \frac{d\delta(t)}{dt} + 3\delta(t) + 5e^{3t} \), where \( \delta(t) \) is the unit impulse.
4.4 Applying the inverse Laplace transform

**Factorization of \( A(s) \)**

The next step is to factorize \( A(s) \) into \( n \) first-order factors

\[
A(s) = (s - p_1)(s - p_2) \ldots (s - p_n) \tag{4.38}
\]

where \( p_k, k = 1,2, \ldots, n \), are the roots of the characteristic equation \( A(s) = 0 \).

Complex roots always appear as pairs of complex-conjugated roots, but otherwise any combination of real and complex roots is possible. The form of the PFE depends on the properties of the roots.

- **\( d \) distinct (unequal) real roots** yield the PFE

  \[
  \sum_{k=1}^{d} \frac{C_k}{(s-p_k)} \tag{4.39}
  \]

  where \( C_k, k = 1,2, \ldots, d \), are constants that need to be determined.

- **\( r \) repeated (equal) real roots** yield the PFE

  \[
  \sum_{k=1}^{r} \frac{C_k}{(s-p_r)^k} \tag{4.40}
  \]

  where \( C_k, k = 1,2, \ldots, r \), are constants that need to be determined and \( p_r = p_k \) are the \( r \) repeated roots. In practice, repeated roots are not very common.
4.4.2 Partial fraction expansion

- **A pair of complex-conjugated roots** \( p = \sigma \pm j\omega \), where \( j = \sqrt{-1} \) is the imaginary unit, yields the PFE

\[
\frac{C_1 s + C_2}{(s - \sigma)^2 + \omega^2}
\]

(4.41)

where \( C_1 \) and \( C_2 \) are constants that need to be determined. Repeated complex-conjugated roots result in rising powers of the numerator similarly as for repeated real roots. In practice, such roots are very rare.

- **The full PFE** for \( d \) distinct real roots \( (p_1, \ldots, p_d) \), \( r \) repeated real roots \( (p_r = p_{d+1} = \cdots = p_{d+r}) \), and one pair of complex-conjugated roots \( (p_{n-1} \text{ and } p_n) \) is

\[
F(s) = F_0(s) + \sum_{k=1}^{d} \frac{c_k}{s - p_k} + \sum_{k=d+1}^{d+r} \frac{c_k}{(s - p_r)^k} + \frac{c_{n-1} s + c_n}{(s - \sigma)^2 + \omega^2}
\]

(4.42)

All terms in the PFE (4.42) are such that their inverse transforms are easily found in the Laplace Transform Table in Section 4.5. According to the superposition theorem, the final function \( f(t) \) is the sum of the individual inverse transforms.

**Note** that the numerator of \( F(s) \) does not affect the form of the PFE.
4.4 Applying the inverse Laplace transform

Calculation of the constants $C_k$

The constants $C_k$, $k = 1, 2, \ldots, n$, can be determined in several ways.

- Because the PFE must apply for any value of the variable $s$, $n$ different, “appropriately” chosen, values of $s$ can be substituted into the PFE to yield $n$ equations, from which the constants $C_k$ can be determined.

- Another, more systematic method is to multiply both sides of the PFE by $A(s)$ and to cancel out the denominators against the same factor in $A(s)$. The expression thus obtain must be equal to $B_0(s)$. The constants $C_k$ can then be determined from the $n$ number of equations that arise when the PFE is required to apply separately for each power of $s$.

- If the roots are distinct and real, $C_k$ can be determined according to

$$C_k = \lim_{s \to p_k} (s - p_k) \frac{B_0(s)}{A(s)} \quad (4.43)$$

Note that the factor $(s - p_k)$ is cancelled out against the same factor in $A(s)$. 
Example 4.7. Ramp response of a first-order system.

The ramp response of a system is the output as a function of time when the input changes as a ramp, i.e., linearly with time. The input is thus

\[ u(t) = bt, \quad t \geq 0 \]  

where \( b \) is the slope of the input ramp. The function has the Laplace transform (pt 2 in the Laplace transform table)

\[ U(s) = b/s^2 \]

(2)

The Laplace-domain output of a first-order system is given by

\[ Y(s) = G(s)U(s), \quad G(s) = \frac{K}{Ts+1} \]

(3)

Substitution of (2) into (3) yields

\[ Y(s) = \frac{Kb}{s^2(Ts+1)} \]

(4)

The time-domain output \( y(t) \) is given by the inverse Laplace transform of (4). This can be found directly in the Laplace Transform Table (pt 27), but here a partial fraction expansion of (4) will be used to find the solution.
The function (4) is **strictly proper** (i.e., no need to do a long division) and the denominator is **already factored**. There is

- a **simple root** \( s = -1/T \), and
- a **double root** \( s = 0 \).

In accordance with (4.42), the PFE is

\[
Y(s) = \frac{Kb}{s^2(Ts+1)} \equiv \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{(Ts+1)}
\]  

(5)

Multiplication of both sides by \( s^2(Ts + 1) \), and cancellation of common factors, yields

\[
Kb \equiv C_1 s(Ts + 1) + C_2 (Ts + 1) + C_3 s^2
\]  

(6)

This expression must apply separately for each power of \( s \):

\[
s^0: \quad Kb = C_2 \quad \Rightarrow \quad C_2 = Kb
\]

\[
s^1: \quad 0 = C_1 + C_2 T \quad \Rightarrow \quad C_1 = -KbT
\]

\[
s^2: \quad 0 = C_1 T + C_3 \quad \Rightarrow \quad C_3 = KbT^2
\]
4.4.2 Partial fraction expansion

Substitution of (7) into (5) gives

\[ Y(s) = \frac{-KbT}{s} + \frac{Kb}{s^2} + \frac{KbT^2}{(Ts+1)} \]  

Points 1, 2 and 25 in the Laplace Transform Table now yield

\[ y(t) = -KbT + Kb t + Kb T e^{-t/T} \]
\[ = Kb(t - T + Te^{-t/T}), \quad t \geq 0 \] (7)

Since \( e^{-t/T} \rightarrow 0 \) as \( t \rightarrow \infty \), the output approaches a ramp with the ramp coefficient \( Kb \).

**Exercise 4.7**

Determine, by the use of partial fraction expansion, the inverse Laplace transform of the following functions:

a) \[ F_a(s) = \frac{s+3}{2s(s-4)^2} \]

b) \[ F_b(s) = \frac{3s+5}{s(s^2+6s+25)} \]
4. Laplace Transform Methods

4.5 Table of Laplace transforms


\[ F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\,dt \quad \Leftrightarrow \quad f(t) = \begin{cases} 0 & \text{für } t < 0 \\ \mathcal{L}^{-1}\{F(s)\} & \text{für } t \geq 0 \end{cases} \]

\[ \mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s) \quad \mathcal{L}\{af(at)\} = F\left(\frac{s}{a}\right) \]

\[ \mathcal{L}\{f'(t)\} = sF(s) - f(0^-) \quad \mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0^-) - f'(0^-) \]

\[ \mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - s f^{(n-2)}(0^-) - f^{(n-1)}(0^-) \]

\[ \mathcal{L}\left[\int_{0}^{t} f(x)\,dx\right] = \frac{F(s)}{s} \quad \mathcal{L}\{e^{at} f(t)\} = F(s - a) \]

\[ \mathcal{L}^{-1}\{e^{-as} F(s)\} = \begin{cases} 0 & \text{für } t < a \\ f(t - a) & \text{für } t \geq a \end{cases} \]

\[ \mathcal{L}\{tf(t)\} = -F'(s) \quad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(\sigma)\,d\sigma \]
4. Laplace Transform Methods

4.5 Table of Laplace transforms

\[ \mathcal{L}\{f(t) \ast g(t)\} = \mathcal{L}\left\{ \int_0^t f(x)g(t-x)dx \right\} = F(s)G(s) \]  
(faltung)

\[ \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt \quad \text{om } f(t) = f(t+T) \]  
(periodisk funktion)

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} [sF(s)] \quad \text{om gränsvärdet existerar} \]  
(slutvärdesteoremet)

\[ f(0^+) = \lim_{s \to \infty} [sF(s)] \]  
(initialvärdesteoremet)

<table>
<thead>
<tr>
<th>( f(t) ), ( t \geq 0 )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( 1 ) (also ( \sigma(t) ))</td>
<td>( 1/s )</td>
</tr>
<tr>
<td>2. ( t )</td>
<td>( 1/s^2 )</td>
</tr>
<tr>
<td>3. ( t^n ), ( n = 0, 1, 2, \ldots )</td>
<td>( n!/s^{n+1} )</td>
</tr>
<tr>
<td>4. ( t^a ), ( a &gt; 0 )</td>
<td>( \Gamma(a+1)/s^{a+1} )</td>
</tr>
<tr>
<td>5. ( \delta(t-a) ) (Impulsfunktionen)</td>
<td>( e^{-as} )</td>
</tr>
<tr>
<td>6. ( \delta(t) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>7. ( u(t-a) = \begin{cases} 0 &amp; \text{före } t &lt; a \ 1 &amp; \text{före } t \geq a \end{cases} ) (Stegfunktionen)</td>
<td>( e^{-as}/s )</td>
</tr>
</tbody>
</table>
### 4.5 Table of Laplace transforms

<table>
<thead>
<tr>
<th>$f(t)$, $t \geq 0$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
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<tbody>
<tr>
<td>8. $e^{-at}$</td>
<td>$\frac{1}{s + a}$</td>
</tr>
<tr>
<td>9. $\frac{1}{a}(1 - e^{-at})$</td>
<td>$\frac{1}{s(s + a)}$</td>
</tr>
<tr>
<td>10. $\frac{1}{a^2}(at - 1 + e^{-at})$</td>
<td>$\frac{1}{s^2(s + a)}$</td>
</tr>
<tr>
<td>11. $te^{-at}$</td>
<td>$\frac{1}{(s + a)^2}$</td>
</tr>
<tr>
<td>12. $\frac{1}{a^2}\left[1 - (1 + at)e^{-at}\right]$</td>
<td>$\frac{1}{s(s+a)^2}$</td>
</tr>
<tr>
<td>13. $\frac{1}{a^2}\left[t - \frac{2}{a} + (\frac{2}{a} + t)e^{-at}\right]$</td>
<td>$\frac{1}{s^2(s+a)^2}$</td>
</tr>
<tr>
<td>14. $(1 - at)e^{-at}$</td>
<td>$\frac{s}{(s + a)^2}$</td>
</tr>
<tr>
<td>15. $\frac{t^ne^{-at}}{n!}$, $n = 0, 1, 2, ...$</td>
<td>$\frac{1}{(s + a)^{n+1}}$</td>
</tr>
<tr>
<td>16. $\frac{t^pe^{-at}}{\Gamma(p+1)}$, $p &gt; 0$</td>
<td>$\frac{1}{(s + a)^{p+1}}$</td>
</tr>
</tbody>
</table>
### 4. Laplace Transform Methods

#### 4.5 Table of Laplace transforms

<table>
<thead>
<tr>
<th>17. $f(t), t \geq 0$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{b-a} (e^{-at} - e^{-bt}), a \neq b$</td>
<td>$\frac{1}{(s+a)(s+b)}$</td>
</tr>
<tr>
<td>$\frac{1}{ab} + \frac{1}{a(a-b)} e^{-at} - \frac{1}{b(a-b)} e^{-bt}$</td>
<td>$\frac{1}{s(s+a)(s+b)}$</td>
</tr>
<tr>
<td>$\frac{abt - a-b}{a^2b^2} - \frac{1}{a^2(a-b)} e^{-at} + \frac{1}{b^2(a-b)} e^{-bt}$</td>
<td>$\frac{1}{s^2(s+a)(s+b)}$</td>
</tr>
<tr>
<td>$\frac{1}{b-a} (be^{-bt} - ae^{-at}), a \neq b$</td>
<td>$\frac{s}{(s+a)(s+b)}$</td>
</tr>
</tbody>
</table>

Formlerna 21-24 gäller om $4b - a^2 > 0$, $D = \sqrt{4b - a^2}$.

| 21. $\frac{2}{D} e^{-\frac{a}{2}t} \sin\left(\frac{D}{2}t\right)$ | $\frac{1}{s^2 + as + b}$ |
| 22. $\frac{1}{b} \left\{ 1 - e^{-\frac{a}{2}t} \left[ \frac{a}{D} \sin\left(\frac{D}{2}t\right) + \cos\left(\frac{D}{2}t\right) \right] \right\}$ | $\frac{1}{s(s^2 + as + b)}$ |
| 23. $\frac{1}{b} \left\{ t - \frac{a}{b} + e^{-\frac{a}{2}t} \left[ \frac{a^2 - 2b}{bD} \sin\left(\frac{D}{2}t\right) + \frac{a}{b} \cos\left(\frac{D}{2}t\right) \right] \right\}$ | $\frac{1}{s^2(s^2 + as + b)}$ |
| $e^{-\frac{a}{2}t} \left[ \cos\left(\frac{D}{2}t\right) - \frac{a}{D} \sin\left(\frac{D}{2}t\right) \right]$ | $\frac{s}{(s^2 + as + b)}$ |
### 4.5 Table of Laplace transforms

<table>
<thead>
<tr>
<th>$f(t)$, $t \geq 0$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{T} e^{-t/T}$</td>
<td>$\frac{1}{Ts + 1}$</td>
</tr>
<tr>
<td>$1 - e^{-t/T}$</td>
<td>$\frac{1}{s(Ts + 1)}$</td>
</tr>
<tr>
<td>$t - T + T e^{-t/T}$</td>
<td>$\frac{1}{s^2 (Ts + 1)}$</td>
</tr>
<tr>
<td>$\frac{1}{T^2} te^{-t/T}$</td>
<td>$\frac{1}{(Ts + 1)^2}$</td>
</tr>
<tr>
<td>$1 - \frac{T + t}{T} e^{-t/T}$</td>
<td>$\frac{1}{s(Ts + 1)^2}$</td>
</tr>
<tr>
<td>$t - 2T + (t + 2T)e^{-t/T}$</td>
<td>$\frac{1}{s^2(Ts + 1)^2}$</td>
</tr>
<tr>
<td>$\frac{T_2}{T_1^2} \left(1 - \frac{t}{T_1} + \frac{t}{T_2}\right) e^{-t/T_1}$</td>
<td>$\frac{T_2s + 1}{(T_1s + 1)^2}$</td>
</tr>
<tr>
<td>$\frac{1}{T_1 - T_2} \left(e^{-t/T_1} - e^{-t/T_2}\right)$</td>
<td>$\frac{1}{(T_1s + 1)(T_2s + 1)}$</td>
</tr>
<tr>
<td>$1 + \frac{1}{T_2 - T_1} (T_1 e^{-t/T_1} - T_2 e^{-t/T_2})$</td>
<td>$\frac{1}{s(T_1s + 1)(T_2s + 1)}$</td>
</tr>
<tr>
<td>$t - T_1 - T_2 + \frac{1}{T_2 - T_1} (T_2^2 e^{-t/T_2} - T_1^2 e^{-t/T_1})$</td>
<td>$\frac{1}{s^2(T_1s + 1)(T_2s + 1)}$</td>
</tr>
</tbody>
</table>
### 4. Laplace Transform Methods

<table>
<thead>
<tr>
<th>$f(t), \ t \geq 0$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$35. \ \frac{T_1 - T_3}{T_1(T_1 - T_2)} e^{-t/T_1} - \frac{T_2 - T_3}{T_2(T_1 - T_2)} e^{-t/T_2}$</td>
<td>$\frac{T_3s + 1}{(T_1s + 1)(T_2s + 1)}$</td>
</tr>
</tbody>
</table>

Formlerna 36–37 gäller om, $0 < \zeta < 1$, $Z = \sqrt{1 - \zeta^2}$

| $36. \ \frac{\omega}{Z} e^{-\zeta \omega t} \sin(\omega Zt)$ | $\frac{1}{1 + 2\zeta s / \omega + s^2 / \omega^2}$ |
| $37a. \ 1 - e^{-\zeta \omega t} \left[ \cos(\omega Zt) + \frac{\zeta}{Z^2} \sin(\omega Zt) \right]$ | $\frac{1}{s(1 + 2\zeta s / \omega + s^2 / \omega^2)}$ |
| $37b. \ 1 - \frac{1}{Z} e^{-\zeta \omega t} \sin(\omega Zt + \phi), \ \phi = \arctan\left(\frac{Z}{\zeta}\right)$ | $\frac{1}{s(1 + 2\zeta s / \omega + s^2 / \omega^2)}$ |

| $38. \ \sin at$ | $\frac{a}{s^2 + a^2}$ |
| $39. \ \cos at$ | $\frac{s}{s^2 + a^2}$ |
| $40. \ \sinh at$ | $\frac{a}{s^2 - a^2}$ |
| $41. \ \cosh at$ | $\frac{s}{s^2 - a^2}$ |
### 4. Laplace Transform Methods

#### 4.5 Table of Laplace transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t), t \geq 0$</td>
<td>$F(s) = \mathcal{L}{f(t)}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expression</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-bt} \sin at$</td>
<td>$\frac{a}{(s+b)^2 + a^2}$</td>
</tr>
<tr>
<td>$e^{-bt} \cos at$</td>
<td>$\frac{s+b}{(s+b)^2 + a^2}$</td>
</tr>
<tr>
<td>$\frac{1}{a^2} (1 - \cos at)$</td>
<td>$\frac{1}{s(s^2 + a^2)}$</td>
</tr>
<tr>
<td>$\frac{1}{a^3} (at - \sin at)$</td>
<td>$\frac{1}{s^2 (s^2 + a^2)}$</td>
</tr>
<tr>
<td>$\frac{1}{2a^3} (\sin at - at \cos at)$</td>
<td>$\frac{1}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$\frac{1}{2a} \sin at$</td>
<td>$\frac{s}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$t \cos at$</td>
<td>$\frac{s^2 - a^2}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$\frac{1}{2a} (\sin at + at \cos at)$</td>
<td>$\frac{s^2}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$\frac{1}{b^2 - a^2} (\cos at - \cos bt)$</td>
<td>$\frac{s}{(s^2 + a^2)(s^2 + b^2)}$</td>
</tr>
</tbody>
</table>
4. Laplace Transform Methods

4.5 Table of Laplace transforms

\[ f(t), \ t \geq 0 \quad \quad F(s) = \mathcal{L}\{f(t)\} \]

51. \[ \frac{\sqrt{a^2 + b^2}}{a} \sin(at + \phi), \]
\[ \phi = \begin{cases} \arctan\left(\frac{a}{b}\right) + 2\pi n, & b > 0 \\ \arctan\left(\frac{a}{b}\right) + \pi + 2\pi n, & b < 0 \end{cases}, \quad n \in \mathbb{Z} \]
\[ \frac{s + b}{s^2 + a^2} \]

52. \[ \frac{\sqrt{a^2 + (b-c)^2}}{a} e^{-ct} \sin(at + \phi), \]
\[ \phi = \begin{cases} \arctan\left(\frac{a}{b-c}\right) + 2\pi n, & b > c \\ \arctan\left(\frac{a}{b-c}\right) + \pi + 2\pi n, & b < c \end{cases}, \quad c > 0, \ n \in \mathbb{Z} \]
\[ \frac{s + b}{(s + c)^2 + a^2} \]

53. \[ \frac{b}{a^2 + c^2} + \frac{\sqrt{a^2 + (b-c)^2}}{a^2 + c^2} e^{-ct} \sin(at + \phi), \]
\[ \phi = \begin{cases} \arctan\left(\frac{a}{b-c}\right) + \arctan\left(\frac{a}{c}\right) + 2\pi n, & b > c \\ \arctan\left(\frac{a}{b-c}\right) + \arctan\left(\frac{a}{c}\right) + \pi + 2\pi n, & b < c \end{cases}, \quad c > 0, \ n \in \mathbb{Z} \]
\[ \frac{s + b}{s\left[(s + c)^2 + a^2\right]} \]