3. Mathematical Modeling

3.2 Models for technical systems

3.2.1 Electrical systems

Fig. 3.1 shows three basic components of electrical circuits.

Variables

- \( t \) = time, \( u \) = voltage [V], \( i \) = current [A]

Component parameters

- \( R \) = resistance [\( \Omega \)], \( C \) = capacitance [F], \( L \) = inductance [H]

Relationships

- Resistance (Ohm’s law): \( u(t) = R \ i(t) \) \hspace{1cm} (3.1)
- Capacitor: \( u(t) = u(0) + \frac{1}{C} \int_{0}^{t} i(\tau) \, d\tau \) \hspace{1cm} (3.2)
- Inductor: \( u(t) = L \frac{di(t)}{dt} \) \hspace{1cm} (3.3)
3.2.1 Electrical systems

- **Example 3.1. A passive analog low-pass filter.**

Figure 3.2 shows a *passive analog low-pass filter*.

- How the voltage $u_{ut}(t)$ on the output side depends on the voltage $u_{in}(t)$ on the input side if the circuit is uncharged at the output?

**Notation:**
- $u_R(t) = $ voltage across the resistor, $i_R(t) = $ current through the resistor
- $u_C(t) = $ voltage across the capacitor, $i_C(t) = $ current through the capacitor

If we count all the voltages (voltage drop) as positives when applying *Kirchhoff's second law* around the right loop, we obtain

$$u_{in}(t) = u_R(t) + u_C(t) \quad (1)$$
$$u_{ut}(t) = u_C(t) \quad (2)$$

When the output is uncharged, there is no current out from the filter, and we have

$$i_R(t) = i_C(t) \quad (3)$$
3.2.1 Electrical systems

The combination of (1) and (2) and substitution with (3.1) gives

\[ u_{ut}(t) = u_{in}(t) - R \cdot i_R(t) \]  \hspace{1cm} (4)

Furthermore, combining (2) and (3.2) gives

\[ u_{ut}(t) = u_C(t) = u_C(0) + \frac{1}{C} \int_0^t i_C(\tau) d\tau \] \hspace{1cm} (5)

The derivative of both sides of (5) with respect to the time gives

\[ \frac{du_{ut}(t)}{dt} = \frac{1}{C} i_C(\tau) = \frac{1}{C} i_R(t) \] \hspace{1cm} (6)

where the last equality is given by (3). Combining (4) and (6) gives

\[ RC \cdot \frac{du_{ut}(t)}{dt} + u_{ut}(t) = u_{in}(t) \] \hspace{1cm} (7)

This is a linear first order differential equation. The circuit is a low-pass filter that filters (i.e., reduces the amplitude of) high frequencies in \( u_{in}(t) \). In practice, we also have an amplifier on the output side, which allows us to charge the circuit so that (3) still holds (approximately).
3.2.1 Electrical systems

- **Example 3.2. Simple RLC circuit.**

  Figure 3.3 shows a simple *RLC circuit* driven by a current source.
  - How does the voltage across the capacitor depend on the current from the current source?

  Notation:
  - $u_R(t) = \text{voltage across the resistor}$, $i_R(t) = \text{current through the resistor}$
  - $u_C(t) = \text{voltage across the capacitor}$, $i_C(t) = \text{current through the capacitor}$
  - $u_L(t) = \text{voltage across the inductor}$, $i_L(t) = \text{current through the inductor}$

  *Kirchhoff’s laws* give

  $$u_C(t) = u_R(t) + u_L(t) \quad (1)$$
  $$i(t) = i_R(t) + i_C(t) \quad (2)$$
  $$i_R(t) = i_L(t) \quad (3)$$
3.2.1 Electrical systems

Substitution of (3.1) and (3.3) in (1):

\[ u_C(t) = R \cdot i_R(t) + L \cdot \frac{di_L}{dt} \]  

(4)

Elimination of \( i_R(t) \) and \( i_L(t) \):

\[ u_C(t) = R \cdot (i(t) - i_C(t)) + L \cdot \frac{d(i(t) - i_C(t))}{dt} \]  

(5)

According to eq. (6) in Ex. 3.1:

\[ i_C(t) = C \cdot \frac{du_C}{dt} \]  

(6)

Substitution of (6) in (5):

\[ u_C(t) = R \cdot \left( i(t) - C \cdot \frac{du_C}{dt} \right) + L \cdot \frac{d\left(i(t) - C \cdot \frac{du_C}{dt}\right)}{dt} \]

After rearrangement:

\[ LC \cdot \frac{d^2u_C}{dt^2} + RC \cdot \frac{du_C}{dt} + u_C(t) = R \cdot i(t) + L \cdot \frac{di}{dt} \]  

(7)

where \( i(t) \) is the input signal and \( u_C(t) \) is the output signal.

This is a linear second order differential equation.
3.2 Model for technical systems

3.2.2 Mechanical systems

The modeling of mechanical systems are mainly based on Newton’s second law

\[ F = ma \]  

(3.4)

where \( F \) is the force acting on the mass \( m \) and \( a \) is the acceleration of the mass.

- Example 3.3. Undamped pendulum.

Figure 3.4 shows an undamped swinging pendulum. The pendulum can only move in two directions in the plane of the figure. Its point of suspension is at a distance \( u \) and its center of mass (the round weight at the lower end of the pendulum) is at a distance \( y \) from the vertical line to the left.

- How does the position \( y \) of the center of mass depend on the position \( u \) of the suspension point?

Notations:
- \( l \) = pendulum’s length, \( \theta \) = angle the pendulum swings away from a vertical position
- \( m \) = weight of mass, \( h \) = vertical position of the center of mass
- \( F \) = force acting in the negative direction on the suspension point of the pendulum
3.2.2 Mechanical systems

When the pendulum is affected by the suspension force \( F \) and the gravitational force \( mg \), according to Newton’s second law, we obtain

horizontal force components: \( m\ddot{y} = -F \sin \theta \) \hspace{1cm} (1)

vertical force components: \( m\ddot{h} = -F \cos \theta + mg \) \hspace{1cm} (2)

\( \ddot{y} \) and \( \ddot{h} \) are second order time derivatives of \( y \) and \( h \), respectively, i.e. the acceleration in the respective directions.

Assume that the pendulum’s swing is moderate so that the angle \( \theta \) is always small. The pendulum then moves hardly at all in the vertical direction and we can assume that \( \ddot{h} \approx 0 \). The elimination of \( F \) then gives

\[ \ddot{y} + g \tan \theta = 0 \] \hspace{1cm} (3)

The angle \( \theta \) is given by the trigonometric identity

\[ \tan \theta = \frac{y-u}{h} \approx \frac{y-u}{l} \] \hspace{1cm} (4)

where the last equality uses the fact that \( h \approx l \) when \( \theta \) is small. By combining (3) and (4) we obtain the model

\[ \ddot{y} + \left( \frac{g}{l} \right) y = \left( \frac{g}{l} \right) u \] \hspace{1cm} (5)

Notice that the approximations \( \ddot{h} \approx 0 \) and “\( \theta \) small” limit the validity of the model.
3.2.2 Mechanical systems

- Example 3.4. Suspension system in an automobile.

**Figure 3.5.** a) Spring-mounted mass with damping; b) Car suspension system.
3.2.2 Mechanical systems

a) How does the vertical deviation $y(t)$ from an equilibrium position depend on a force $u(t)$ acting on the spring-mounted mass $m$?

An equilibrium position applies when $y = u = 0$ (apart from the units). If the downward direction is the positive vertical direction, Newton’s second law for the spring force and the damping force of the cylinder gives

$$m\ddot{y} = -ky - b\dot{y} + u(t) \quad \text{i.e.} \quad m\ddot{y} + b\dot{y} + ky = u(t) \quad (1)$$

where $b$ and $k$ are constants. The gravitational force $mg$ is not included; because it also affects the equilibrium position, it is cancelled out when the deviation from the equilibrium position is modeled.

b) How do the deviations $y_1(t)$ and $y_2(t)$ in a car suspension system depend on $u(t)$, which denotes the roughness of the ground? $m_1$ is the mass of the car, $m_2$ is the mass of the wheels and the axles, $b_1$ and $k_1$ describe the dynamics of the car shock absorber and $k_2$ denotes the elasticity of the tires. In the equilibrium position, $y_1 = y_2 = u = 0$. If the upward direction is the positive direction, we get

$$m_1\ddot{y}_1 = k_1(y_2 - y_1) + b_1(\dot{y}_2 - \dot{y}_1) \quad (2)$$

$$m_2\ddot{y}_2 = k_1(y_1 - y_2) + b_1(\dot{y}_1 - \dot{y}_2) + k_1(u - y_2) \quad (3)$$

These are two coupled 2nd order differential equations, that describe the car body and the vertical motion of the wheels as function of the vertical roughness of the road.
3.2 Model for technical systems

3.2.3 Process engineering systems

Process engineering systems are typically modeled with flow balances (mass and energy balances) and constitutive relationships.

- **Example 3.5. Liquid container with free outflow.**

A volumetric flow rate \( u \) is supplied continuously to the container and a volumetric flow rate \( q \) flows out freely by gravity, caused by the height of the liquid \( h \) in the container. The container has a constant cross-sectional area \( A \), and the outlet tube has the “effective” cross-sectional area \( a \).

- How does the level of the liquid depend on the inflow \( u \)?

We assume that the liquid has constant density \( \rho \).

**Mass balance:**

\[
\frac{d}{dt}(\rho Ah) = \rho u - \rho q \tag{1}
\]

Because the density and the cross-sectional area are constant, (1) can be simplified to

\[
A \frac{dh}{dt} = u - q \tag{2}
\]
3.2.3 Process engineering systems

According to Bernoulli’s law, the following constitutive relationship applies for the outflow of a liquid

\[ v = \sqrt{2gh} \]  \hspace{1cm} (3)

where \( v \) is the velocity of the outflow and \( g \) is the acceleration of gravity. Due to the contraction at the beginning of the outflow tube, the volume flow rate \( q \) is then given by

\[ q = av = a\sqrt{2gh} \]  \hspace{1cm} (4)

where \( a \) is the effective cross-sectional area of the outflow tube, which is slightly smaller than the actual cross-section. Combination of (2) and (4) finally gives

\[ \frac{dh}{dt} = -\frac{a\sqrt{2g}}{A} \sqrt{h} + \frac{1}{A}u \]  \hspace{1cm} (5)

i.e. a nonlinear differential equation that describes how the level \( h \) depends on the inflow \( u \).
3.2.3 Process engineering systems

- **Example 3.6. Mixing tank.**

Two volumetric flow rates \( F_1 \) and \( F_2 \), with the concentrations (mass/volume) \( c_1 \) and \( c_2 \), respectively, of some inflowing component X. They are mixed continuously in a container and a volumetric flow rate \( F_3 \), with concentration \( c_3 \), is discharged from the container. The liquid in the container, which has a constant cross-sectional area \( A \), reaches the height \( h \). The concentration in the container of the component X is \( c \). *The stirring in the container is assumed to be perfect.*

- How do the level \( h \) and the concentration \( c \) (and \( c_3 \)) depend on other variables?

It is reasonable to assume that the liquid density in the different flows is constant and the same if the liquid temperature is constant and the concentration of the components is moderate. Analogously to ex. 3.5, we obtain after cancelling out the density

**Total mass balance:**

\[
A \frac{dh}{dt} = F_1 + F_2 - F_3
\]  

(1)

We cannot eliminate the outflow \( F_3 \) because we do not know what it depends on.
3.2.3 Process engineering systems

We can also set up a mass balance for each component $X$, 

**partial mass balance:**

$$ \frac{d}{dt} (Ahc) = F_1c_1 + F_2c_2 - F_3c_3 \tag{2} $$

If the stirring in the container is perfect, we have *complete mixing* which means that the concentration is the same all over the container at a given time instant. This also means that the concentration in the outflow must be the same as the concentration in the container, i.e. we have *the constitutive relationship*

$$ c_3 = c \tag{3} $$

The development of the derivative in (2) according to the product rule and considering equation (3) gives

$$ Ac \frac{dh}{dt} + Ah \frac{dc}{dt} = F_1c_1 + F_2c_2 - F_3c \tag{4} $$

Then, combination of (4) with (1) gives

$$ Ah \frac{dc}{dt} = F_1(c_1 - c) + F_2(c_2 - c) \tag{5} $$

This is a linear differential equation with (in general) non-constant parameters.
3.2.3 Process engineering systems

Example 3.7. Water-heater.

The inflow of water is a mass flow $\dot{m}_1$ with temperature $T_1$ and the outflow is a mass flow $\dot{m}_1$ with temperature $T_2$. The water, with mass $M$ in the heater, is heated up to a temperature $T$ with a heating flow rate $Q$: The stirring in the heater is assumed to be perfect.

– How do the water volume and the temperature in the heater depend on other variables?

Mass balance:

$$\frac{dM}{dt} = \dot{m}_1 - \dot{m}_2$$  \hspace{1cm} (1)

Energy balance:

$$\frac{dE}{dt} = \dot{E}_1 - \dot{E}_2 + \dot{Q}$$  \hspace{1cm} (2)

where $\dot{E}_1$ and $\dot{E}_2$ are energy flows that are supplied by the inflow and the outflow, respectively.
3.2.3 Process engineering systems

The energy in a substance is proportional to its mass or mass flow rate, and for liquids it applies with good accuracy that the energy is also proportional to the temperature. This gives

**Constitutive relationships:**

\[ E = c_p T M, \quad \dot{E}_1 = c_p T_1 \dot{m}_1, \quad \dot{E}_2 = c_p T_2 \dot{m}_2 \]  

(3)

where \( c_p \) is the **specific heat capacity** for water (in this case assumed to be constant). Combining (2) and (3) and the development of the derivative according to the chain rule gives

\[ T \frac{dM}{dt} + M \frac{dT}{dt} = T_1 \dot{m}_1 - T_2 \dot{m}_2 + \frac{\dot{Q}}{c_p} \]  

(4)

The **constitutive relationship** \( T_2 = T \) applies based on the assumption of **perfect mixing**.

The elimination of \( dM / dt \) with (1) gives

\[ M \frac{dT}{dt} = \dot{m}_1 (T_1 - T) + \frac{\dot{Q}}{c_p} \]  

(5)

Equation (1) and (5) indicate how the mass and the temperature in the heater depend on the inflow and the heating efficiency \( \dot{Q} \).
3.2.3 Process engineering systems

If we want to use units of volume instead of units of mass, we can now insert $M = \rho Ah$ and $m_1 = \rho_1 F_1$ in equation (5) to get

$$
\rho Ah \frac{dT}{dt} = \rho_1 F_1 (T_1 - T) + \dot{Q} \frac{1}{c_p}
$$

(6)

Note that in equation (6) the density is not assumed to be a constant.

Equation (1) expressed in units of volume becomes more complex when we have a variable density. We can, however, show that even if the temperature dependence of the density is not negligible, the effects of a variable density on (1) tend to cancel out. A completely adequate form of (1) expressed in units of volume is then

$$
A \frac{dh}{dt} = F_1 - F_2
$$

(7)
Example 3.8. Gas in a closed tank.

Figure 3.9 illustrates a closed gas-tank with the volume $V$, substance amount (molar amount) $n$, pressure $p$ and temperature $T$. The inflow to the tank is the molar flow rate $\dot{n}_1$ at the pressure $p_1$, the outflow is the molar flow rate $\dot{n}_2$ at the pressure $p_2$. Valve 2 can be used for control by adjusting the valve position $u$.

- How does the pressure $p$ in the tank depend on other variables?

Substance amount balance:

$$\frac{dn}{dt} = \dot{n}_1 - \dot{n}_2$$  \hspace{1cm} (1)

The molar flow rate through a valve with a given opening position is proportional to the square root of the pressure difference across the valve. Moreover, it is assumed that the factor of proportionality is proportional to the square of the “linear” valve position $u$. The molar flow rates are then given by

constitutive relationships:

$$\dot{n}_1 = k_1 \sqrt{p_1 - p} \quad , \quad \dot{n}_2 = k_2 u^2 \sqrt{p - p_2}$$  \hspace{1cm} (2)
3.2.3 Process engineering systems

Furthermore, we can assume that the ideal gas law holds, i.e.

\[ pV = nRT \]  \hspace{1cm} (3)

applies. Here \( R \) is the general gas constant and \( T \) is the temperature expressed in Kelvin.

If the temperature \( T \) is constant, then substitution of (2) and (3) in (1) gives

\[ \frac{dp}{dt} = \frac{RT}{V} \frac{dn}{dt} = \frac{RT}{V} \left( k_1 \sqrt{p_1 - p} - k_2 u^2 \sqrt{p - p_2} \right) \]  \hspace{1cm} (4)

which is a relatively complex nonlinear differential equation, even if it is of first order.