3. Mathematical Modelling

3.1 Modelling principles
   3.1.1 Model types
   3.1.2 Model construction
   3.1.3 Modelling from first principles

3.2 Models for technical systems
   3.2.1 Electrical systems
   3.2.2 Mechanical systems
   3.2.3 Process engineering systems

3.3 Model linearization
   3.3.1 Motivation
   3.3.2 Linearization of ODEs
3. Mathematical Modelling

3.1 Modelling principles

3.1.1 Model types

For design and analysis of a control system we need a mathematical model that describes the dynamical behaviour of the system. The dynamics can be described by

- differential equations for continuous-time dynamics
- difference equations for discrete-time dynamics

Most processes are time continuous, but

- some processes are inherently time discrete (e.g. radioactive decay)
- computer algorithms (i.e. controllers) and many measuring devices produce outputs at discrete time instants

» to design such controllers, we sometimes use discrete-time models to describe continuous-time processes

In this course, we will consider

- both types of models
- controller design both in continuous and discrete time

However, the major part of the course deals with continuous time.
3. Mathematical Modelling

3.1 Modelling principles

3.1.2 Model construction

There are two main principles for construction of mathematical models:

- **Modelling from first principles**: we derive models using physical laws and other known relationships (models).

- **System identification**: we use observations (measurements) of the system to find a model empirically. Usually, designed identification experiments are carried out to generate suitable data.

Often both methods are combined: we derive the basic model from first principles and determine uncertain parameters by system identification.

It is important to realize that **all models have a limited validity range**, even the physical laws (e.g. Newton’s laws of motion do not apply close to the speed of light). It is especially important to note that:

- models determined through system identification should not be used outside the experimental range.
3.1.3 Modelling from first principles

In the following we consider modelling from first principles. Because real technical systems tend to be complex we cannot or do not want to include all details of the system in the model.

We try to make a good compromise between the following two requirements. The model should be

- sufficiently accurate for its intended purpose
- simple enough to use e.g. for system analysis and control design

In modelling from first principles, two types of mathematical relationships are used:

- conservation laws
- constitutive relationships
Conservation laws apply to additive quantities of the same type in a system. There are two general kinds of conservation laws:

- **flow balances**
- **"effort" balances**

A **flow balance** for a given quantity in a system has the general form

\[
\text{accumulation / time unit} = \text{inflow} - \text{outflow} + \text{production / time unit}
\]

where

- accumulation and production occurs inside the system
- inflow and outflow cross the system boundaries

Flow balances apply to conserved quantities (under normal conditions). If no chemical or nuclear reactions take place, the production is zero.

**Examples** of flow balance quantities:

- mass
- particles (moles)
- energy
- current (Kirchhoff’s first law)

**Note** that volume is not a conserved quantity for compressive fluids.
3.1 Modelling principles

3.1.3 Modelling from first principles

An **effort balance** for a given quantity has the general form

\[
\frac{\text{change}}{\text{time unit}} = \text{forcing quantity} - \text{loading quantity}
\]

where

- change refers to a system property
- driving and loading refer to interaction with the surrounding

Generally, effort balances are applications of *Newton’s laws of motion* and *Kirchhoff’s second law*.

Examples of effort balance quantities:

- force
- momentum
- angular momentum
- voltage (Kirchhoff’s second law)
Constitutive relationships

Constitutive relationships are *static relationships* that relate *quantities of different kinds* in a system.

Examples of constitutive relationships:

- **Ohm’s law:** relates the current to the voltage over a resistance
- **valve characteristics:** relates the flow rate to the pressure drop over a valve
- **Bernoulli’s law:** relates the velocity of the flow out of a tank to the liquid level in the tank
- **the ideal gas law:** relates the temperature to the pressure of a gas in a closed container
3.1 Modelling principles

3.1.3 Modelling from first principles

The general modelling procedure

1. Formulate balance equations.
2. Introduce constitutive relationships to
   – relate variables to each other;
   – possibly to introduce new variables in the balance equations.
3. Do a correctness check by at least checking that
   – all additive terms in an equation have the same unit;
   – the left and right hand side of an equation have the same unit.
3. Mathematical Modelling

3.2 Models for technical systems

3.2.1 Electrical systems

Fig. 3.1 shows three basic components of electrical circuits.

Variables
- $t =$ time, $u =$ voltage [V], $i =$ current [A]

Component parameters
- $R =$ resistance [$\Omega$], $C =$ capacitance [F], $L =$ inductance [H]

Relationships
- Resistor (Ohm’s law):
  \[ u(t) = R \, i(t) \]  \hspace{1cm} (3.1)
- Capacitor:
  \[ u(t) = u(0) + \frac{1}{C} \int_0^t i(\tau) \, d\tau \]  \hspace{1cm} (3.2)
- Inductor:
  \[ u(t) = L \frac{di(t)}{dt} \]  \hspace{1cm} (3.3)
3.2 Models for technical systems

3.2.1 Electrical systems

Example 3.1. A passive analog low-pass filter.

Figure 3.2 shows a passive analog low-pass filter.

- How does the voltage $u_{ut}(t)$ on the output side depend on the voltage $u_{in}(t)$ on the input side if the circuit is uncharged at the output?

Notation:
- $u_R(t) =$ voltage across the resistor, $i_R(t) =$ current through the resistor
- $u_C(t) =$ voltage across the capacitor, $i_C(t) =$ current through the capacitor

According to Kirchhoff’s second law, the voltages over the components satisfy

$$u_{in}(t) = u_R(t) + u_C(t) \quad (1)$$

$$u_{ut}(t) = u_C(t) \quad (2)$$

When the output is uncharged, there is no current out from the filter, and we have

$$i_R(t) = i_C(t) \quad (3)$$
3.2.1 Electrical systems

Example 3.1. A passive analog low-pass filter

The combination of (1) and (2) and substitution of (3.1) give

\[ u_{ut}(t) = u_{in}(t) - R \cdot i_R(t) \quad (4) \]

Furthermore, combining (2) and (3.2) gives

\[ u_{ut}(t) = u_C(t) = u_C(0) + \frac{1}{C} \int_0^t i_C(\tau)d\tau \quad (5) \]

The derivative of both sides of (5) with respect to the time gives

\[ \frac{du_{ut}(t)}{dt} = \frac{1}{C} i_C(\tau) = \frac{1}{C} i_R(t) \quad (6) \]

where the last equality is given by (3). Combining (4) and (6) gives

\[ RC \cdot \frac{du_{ut}(t)}{dt} + u_{ut}(t) = u_{in}(t) \quad (7) \]

This is a linear first-order differential equation. The circuit is a low-pass filter that filters (i.e., reduces the amplitude of) high frequencies in \( u_{in}(t) \). In practice, we also have an amplifier on the output side, which allows us to charge the circuit so that (3) still holds (approximately).
3.2 Models for technical systems

3.2.1 Electrical systems

Example 3.2. Simple RLC circuit.

Figure 3.3 shows a simple **RLC circuit** charged by a current source.

- How does the voltage across the capacitor depend on the current from the current source?

![Simple RLC circuit](image)

**Figure 3.3. Simple RLC circuit.**

Notation:

- \( u_R(t) \) = voltage across the resistor, \( i_R(t) \) = current through the resistor
- \( u_C(t) \) = voltage across the capacitor, \( i_C(t) \) = current through the capacitor
- \( u_L(t) \) = voltage across the inductor, \( i_L(t) \) = current through the inductor

**Kirchhoff’s laws** give

\[
\begin{align*}
    u_C(t) &= u_R(t) + u_L(t) \\
    i(t) &= i_R(t) + i_C(t) \\
    i_R(t) &= i_L(t)
\end{align*}
\] (1) (2) (3)
3.2.1 Electrical systems

Example 3.2. Simple RLC circuit

Substitution of (3.1) and (3.3) into (1):

\[ u_C(t) = R \cdot i_R(t) + L \cdot \frac{di_L}{dt} \]  \hspace{1cm} (4)

Elimination of \( i_R(t) \) and \( i_L(t) \):

\[ u_C(t) = R \cdot (i(t) - i_C(t)) + L \cdot \frac{d(i(t) - i_C(t))}{dt} \]  \hspace{1cm} (5)

According to eq. (6) in Ex. 3.1:

\[ i_C(t) = C \cdot \frac{du_C}{dt} \]  \hspace{1cm} (6)

Substitution of (6) into (5):

\[ u_C(t) = R \cdot \left( i(t) - C \cdot \frac{du_C}{dt} \right) + L \cdot \frac{d\left( i(t) - C \cdot \frac{du_C}{dt} \right)}{dt} \]

After rearrangement:

\[ LC \cdot \frac{d^2u_C}{dt^2} + RC \cdot \frac{du_C}{dt} + u_C(t) = R \cdot i(t) + L \cdot \frac{di}{dt} \]  \hspace{1cm} (7)

where \( i(t) \) is the input signal and \( u_C(t) \) is the output signal.

This is a *linear second-order differential equation*.
3.2.2 Mechanical systems

The modeling of mechanical systems are mainly based on Newton’s second law

\[ F = ma \]  \hspace{1cm} (3.4)

where \( F \) is the force acting on the mass \( m \) and \( a \) is the acceleration of the mass.

Example 3.3. Undamped pendulum.

Figure 3.4 shows an undamped swinging pendulum. The pendulum can only move in two directions in the plane of the figure. Its point of suspension is at a distance \( u \) and its center of mass (the round weight at the lower end of the pendulum) is at a distance \( y \) from the vertical line to the left.

- How does the position \( y \) of the center of mass depend on the position \( u \) of the suspension point?

Notation

- \( l \) = pendulum’s length, \( \theta \) = angle the pendulum swings away from a vertical position
- \( m \) = weight of mass, \( h \) = vertical position of the center of mass
- \( F \) = force acting in the negative direction on the suspension point of the pendulum
3.2 Models for technical systems

When the pendulum is affected by the suspension force $F$ and the gravitational force $mg$, according to Newton’s second law, we obtain

horizontal force components: \[ m\ddot{y} = -F \sin \theta \] (1)

vertical force components: \[ m\ddot{h} = -F \cos \theta + mg \] (2)

$\ddot{y}$ and $\ddot{h}$ are second-order time derivatives of $y$ and $h$, respectively, i.e. the acceleration in the respective directions.

Assume that the pendulum’s swing is moderate so that the angle $\theta$ is always small. The pendulum then moves hardly at all in the vertical direction and we can assume that $\ddot{h} \approx 0$. The elimination of $F$ then gives

\[ \ddot{y} + g \tan \theta = 0 \] (3)

The angle $\theta$ is given by the trigonometric identity

\[ \tan \theta = \frac{\dot{y} - \dot{u}}{\dot{h}} \approx \frac{\dot{y} - \dot{u}}{l} \] (4)

where the last equality uses the fact that $h \approx l$ when $\theta$ is small. By combining (3) and (4) we obtain the model

\[ \ddot{y} + \left( \frac{g}{l} \right) y = \left( \frac{g}{l} \right) u \] (5)

Notice that the approximations $\ddot{h} \approx 0$ and “$\theta$ small” limit the validity of the model.
Example 3.4. Suspension system in a car.

Figure 3.5. a) Spring-mounted mass with damping; b) Car suspension system.
3.2.2 Mechanical systems

Example 3.4. Suspension system in a car

a) How does the vertical deviation \( y(t) \) from an equilibrium position depend on a force \( u(t) \) acting on the spring-mounted mass \( m \)?

An equilibrium position applies when \( y = u = 0 \) (apart from the units). If the downward direction is the positive vertical direction, Newton’s second law for the spring force and the damping force of the cylinder gives

\[
m\ddot{y} = -b\dot{y} - ky + u(t) \quad \text{i.e.} \quad m\ddot{y} + b\dot{y} + ky = u(t)
\]  

(1)

where \( b \) and \( k \) are constants. The gravitational force \( mg \) is not included; because it also affects the equilibrium position, it is cancelled out when the deviation from the equilibrium position is modeled.

b) How do the deviations \( y_1(t) \) and \( y_2(t) \) in a car suspension system depend on \( u(t) \), which denotes the roughness of the ground? \( m_1 \) is the mass of the car, \( m_2 \) is the mass of the wheels and the axles, \( b_1 \) and \( k_1 \) describe the dynamics of the car shock absorber and \( k_2 \) denotes the elasticity of the tires. In the equilibrium position, \( y_1 = y_2 = u = 0 \). If the upward direction is the positive direction, we get

\[
m_1\ddot{y}_1 = k_1(y_2 - y_1) + b_1(\dot{y}_2 - \dot{y}_1) \quad (2)
\]

\[
m_2\ddot{y}_2 = k_1(y_1 - y_2) + b_1(\dot{y}_1 - \dot{y}_2) + k_1(u - y_2) \quad (3)
\]

These are two coupled 2nd order differential equations, that describe the car body and the vertical motion of the wheels as function of the vertical roughness of the road.
### 3.2.3 Process engineering systems

Process engineering systems are typically modeled with *flow balances* (mass and energy balances) and *constitutive relationships*.

#### Example 3.5. Liquid container with free outflow.

A volumetric flow rate $u$ is supplied continuously to the container and a volumetric flow rate $q$ flows out freely by gravity, caused by the height of the liquid $h$ in the container. The container has a constant cross-sectional area $A$, and the outlet tube has the “effective” cross-sectional area $a$.

- How does the level of the liquid depend on the inflow $u$?

We assume that the liquid has constant density $\rho$.

**Mass balance:**

$$\frac{d}{dt}(\rho Ah) = \rho u - \rho q$$  \hspace{1cm} (1)

Because the density and the cross-sectional area are constant, (1) can be simplified to

$$A \frac{dh}{dt} = u - q$$  \hspace{1cm} (2)
3.2.3 Process engineering systems

Ex. 3.5. Liquid tank with free outflow

According to Bernoulli’s law, the following constitutive relationship applies for the outflow of a liquid

\[ v = \sqrt{2gh} \quad (3) \]

where \( v \) is the velocity of the outflow and \( g \) is the acceleration of gravity. Due to the contraction at the beginning of the outflow tube, the volume flow rate \( q \) is given by

\[ q = av = a\sqrt{2gh} \quad (4) \]

where \( a \) is the effective cross-sectional area of the outflow tube, which is slightly smaller than the actual cross-section. Combination of (2) and (4) finally gives

\[ \frac{dh}{dt} = -\frac{a\sqrt{2g}}{A} \sqrt{h} + \frac{1}{A} u \quad (5) \]

i.e. a nonlinear differential equation that describes how the level \( h \) depends on the inflow \( u \).
3.2 Models for technical systems

**Example 3.6. Mixing tank.**

Two volumetric flow rates $F_1$ and $F_2$, with the concentrations (mass/volume) $c_1$ and $c_2$, respectively, of some inflowing component $X$. They are mixed continuously in a container and a volumetric flow rate $F_3$, with concentration $c_3$, is discharged from the container. The liquid in the container, which has a constant cross-sectional area $A$, reaches the height $h$. The concentration in the container of the component $X$ is $c$. *The stirring in the container is assumed to be perfect.*

- How do the level $h$ and the concentration $c$ (and $c_3$) depend on other variables?

It is reasonable to assume that the liquid density in the different flows is constant and the same if the liquid temperature is constant and the concentration of the components is moderate. Analogously to Ex. 3.5, we obtain after cancelling out the density

\[
A \frac{dh}{dt} = F_1 + F_2 - F_3 \quad (1)
\]

We cannot eliminate the outflow $F_3$ because we do not know what it depends on.

---

**Figure 3.7. Mixing tank.**
3.2.3 Process engineering systems

We can also set up a mass balance for each component $X$,

**partial mass balance:**

$$\frac{d}{dt}(Ahc) = F_1c_1 + F_2c_2 - F_3c_3$$  \hspace{1cm} (2)

If the stirring in the container is perfect, we have *complete mixing* which means that the concentration is the same all over the container at a given time instant. This also means that the concentration in the outflow must be the same as the concentration in the container, i.e. we have *the constitutive relationship*

$$c_3 = c$$  \hspace{1cm} (3)

The development of the derivative in (2) according to the product rule and considering equation (3) gives

$$Ac \frac{dh}{dt} + Ah \frac{dc}{dt} = F_1c_1 + F_2c_2 - F_3c$$  \hspace{1cm} (4)

Then, combination of (4) with (1) gives

$$Ah \frac{dc}{dt} = F_1(c_1 - c) + F_2(c_2 - c)$$  \hspace{1cm} (5)

This is a **linear differential equation** with (in general) **non-constant parameters**.

Example 3.6. Mixing tank
### Example 3.7. Water heater.

The inflow of water is a mass flow $\dot{m}_1$ with temperature $T_1$ and the outflow is a mass flow $\dot{m}_2$ with temperature $T_2$. The water, with mass $M$ in the heater, is heated up to a temperature $T$ with a heating flow rate $Q$. The stirring in the heater is assumed to be perfect.

- How do the water volume and the temperature in the heater depend on other variables?

**Mass balance:**

$$\frac{dM}{dt} = \dot{m}_1 - \dot{m}_2 \tag{1}$$

**Energy balance:**

$$\frac{dE}{dt} = \dot{E}_1 - \dot{E}_2 + \dot{Q} \tag{2}$$

where $\dot{E}_1$ and $\dot{E}_2$ are energy flows that are supplied by the inflow and the outflow, respectively.
3.2.3 Process engineering systems

The energy in a substance is proportional to its mass or mass flow rate, and for liquids it applies with good accuracy that the energy is also proportional to the temperature. This gives

**Constitutive relationships:**

\[ E = c_p T M, \quad \dot{E}_1 = c_p T_1 \dot{m}_1, \quad \dot{E}_2 = c_p T_2 \dot{m}_2 \]  

(3)

where \( c_p \) is the specific heat capacity for water (in this case assumed to be constant).

Combining (2) and (3) and the development of the derivative according to the chain rule gives

\[
T \frac{dM}{dt} + M \frac{dT}{dt} = T_1 \dot{m}_1 - T_2 \dot{m}_2 + \frac{\dot{Q}}{c_p}
\]

(4)

The constitutive relationship \( T_2 = T \) applies based on the assumption of perfect mixing. The elimination of \( dM / dt \) with (1) gives

\[
M \frac{dT}{dt} = \dot{m}_1 (T_1 - T) + \frac{\dot{Q}}{c_p}
\]

(5)

Equation (1) and (5) indicate how the mass and the temperature in the heater depend on the inflow and the heating efficiency \( \dot{Q} \).
3.2.3 Process engineering systems

Example 3.7 Water heater

If we want to use units of volume instead of units of mass, we can now insert $M = \rho Ah$ and $\dot{m}_1 = \rho_1 F_1$ in equation (5) to get

$$\rho Ah \frac{dT}{dt} = \rho_1 F_1 (T_1 - T) + \frac{\dot{Q}}{c_p}$$

(6)

Note that in equation (6) the density is not assumed to be a constant.

Equation (1) expressed in units of volume becomes more complex when we have a variable density. We can, however, show that even if the temperature dependence of the density is not negligible, the effects of a variable density on (1) tend to cancel out. A completely adequate form of (1) expressed in units of volume is then

$$A \frac{dh}{dt} = F_1 - F_2$$

(7)
### Example 3.8. Gas in a closed tank.

Figure 3.9 illustrates a closed gas-tank with the volume $V$, substance amount (molar amount) $n$, pressure $p$ and temperature $T$. The inflow to the tank is the molar flow rate $\dot{n}_1$ at the pressure $p_1$, the outflow is the molar flow rate $\dot{n}_2$ at the pressure $p_2$. Valve 2 can be used for control by adjusting the valve position $u$.

- How does the pressure $p$ in the tank depend on other variables?

**Molar balance:**

$$\frac{dn}{dt} = \dot{n}_1 - \dot{n}_2$$  \hspace{1cm} (1)

The molar flow rate through a valve with a given opening position is proportional to the square root of the pressure difference across the valve. Moreover, it is assumed that the factor of proportionality is proportional to the square of the “linear” valve position $u$. The molar flow rates are then given by

**constitutive relationships:**

$$\dot{n}_1 = k_1 \sqrt{p_1 - p}, \quad \dot{n}_2 = k_2 u^2 \sqrt{p - p_2}$$  \hspace{1cm} (2)
3.2.3 Process engineering systems

Furthermore, we can assume that the ideal gas law holds, i.e.

\[ pV = nRT \]  \hspace{1cm} (3)

applies. Here \( R \) is the general gas constant and \( T \) is the temperature expressed in Kelvin. If the temperature \( T \) is constant, then substitution of (2) and (3) in (1) gives

\[
\frac{dp}{dt} = \frac{RT}{V} \frac{dn}{dt} = \frac{RT}{V} \left( k_1 \sqrt{p_1 - p} - k_2 u^2 \sqrt{p - p_2} \right) \]  \hspace{1cm} (4)

which is a relatively complex nonlinear differential equation, even if it is of first order.
3. Mathematical Modelling

3.3 Model linearization

3.3.1 Motivation

We have in a number of examples shown how to derive dynamic models for many types of technical systems. In all cases, the obtained models are ordinary differential equations. We note that

- the differential equations (DEs) are often nonlinear
- even if they are linear, the coefficients are generally not constant because they depend on some physical time-varying quantity
- it is difficult, maybe impossible, to find general solutions to these kinds of DEs

Therefore, we need to

- study special cases and/or
- do simplifying assumptions

Frequently used simplifications is to assume that

- some quantities are constant, even if they are (slightly) time-varying
- input signals change in some ideal (but reasonable) way
3.3 Model linearization

In practice, it is often enough to know the system behaviour in a limited region close to a known operating point. Then, the model simplification may be to

- linearize the model equations at the operating point.

The advantage of this is that

- efficient analysis, synthesis, and design methods based on linear algebra can be used.

If the system is very nonlinear, or the operating region very large, one can use

- several linear models that are linearized at different operating points.

Because of the reasons mentioned above,

- modelling from first principles is often followed by a linearization of the model.

In this course we are only considering models obtained from ordinary differential equations, not partial DEs.
3.3.2 Linearization of ODEs

A general ODE

Consider an \( n \):th order ODE, which we can formally write as

\[
f(y^{(n)}, \ldots, \dot{y}, y, u) = 0
\]  
(3.5)

- To simplify, time derivatives of \( u \) are not included; they can be handled in the same way as the time derivatives of \( y \).
- Usually the time derivatives appear linearly in (3.5); however, the linearization applies also when they do not appear linearly.

We can linearize (3.5) by means of a first-order Taylor series expansion at the nominal operating point \((\bar{y}^{(n)}, \ldots, \dot{\bar{y}}, \bar{y}, \bar{u})\), denoted by \( \bar{f} \):

\[
f(y^{(n)}, \ldots, \dot{y}, y, u) \approx f(\bar{y}^{(n)}, \ldots, \dot{\bar{y}}, \bar{y}, \bar{u}) + \left( \frac{\partial f}{\partial y^{(n)}} \right)_{\bar{f}} (y^{(n)} - \bar{y}^{(n)}) + \]
\[
\cdots + \left( \frac{\partial f}{\partial \dot{y}} \right)_{\bar{f}} (\dot{y} - \dot{\bar{y}}) + \left( \frac{\partial f}{\partial y} \right)_{\bar{f}} (y - \bar{y}) + \left( \frac{\partial f}{\partial u} \right)_{\bar{f}} (u - \bar{u})
\]
(3.6)

- Usually the operating point is a static (steady-state) point with all time derivatives equal to zero, but (3.6) holds even if this is not so.
3.3 Model linearization

We introduce the variables

\[ \Delta y^{(n)} \equiv y^{(n)} - \bar{y}^{(n)}, \ldots, \Delta \dot{y} \equiv \dot{y} - \bar{\dot{y}}, \Delta y \equiv y - \bar{y}, \Delta u \equiv u - \bar{u} \quad (3.7) \]

which denote deviations from the nominal operating point. We call such variables *deviation variables*, or simply, \( \Delta \)-variables.

Combination of (3.5), (3.6) and (3.7) with the fact that the operating point satisfies (3.5), gives

\[ \left( \frac{\partial f}{\partial y^{(n)}} \right) \Delta y^{(n)} + \cdots + \left( \frac{\partial f}{\partial \dot{y}} \right) \Delta \dot{y} + \left( \frac{\partial f}{\partial y} \right) \Delta y + \left( \frac{\partial f}{\partial u} \right) \Delta u = 0 \quad (3.8) \]

This is a linear \( n \):th order ODE with constant coefficients.

**Note:** If the ODE contains time derivatives of \( u \), they appear as \( \Delta \)-variables in (3.8) in the same way as the time derivatives of \( y \) appear.
3.3 Model linearization

ODEs linear in time derivatives

If the time derivatives appear linearly in (3.5), we can formally write

\[ f_n(y, u)y^{(n)} + \cdots + f_1(y, u)y + f_0(y, u) = 0 \tag{3.9} \]

We can apply (3.6) to linearize every term separately. We get

\[ f_0(y, u) \equiv f_0(\bar{y}, \bar{u}) + \left( \frac{\partial f_0}{\partial y} \right)_{\bar{f}_0} \Delta y + \left( \frac{\partial f_0}{\partial u} \right)_{\bar{f}_0} \Delta u \]

and for the term with the \( i \):th derivative:

\[ f_i(y, u)y^{(i)} \equiv f_i(\bar{y}, \bar{u})\bar{y}^{(i)} + f_i(\bar{y}, \bar{u})\Delta y^{(i)} \]

\[ + \left( \frac{\partial f_i}{\partial y} \right)_{\bar{f}_i} \bar{y}^{(i)} \Delta y + \left( \frac{\partial f_i}{\partial u} \right)_{\bar{f}_i} \bar{y}^{(i)} \Delta u \]

Substitution into (3.9), which also holds for the operating point, gives

\[ f_n(\bar{y}, \bar{u})\Delta y^{(n)} + \cdots + f_1(\bar{y}, \bar{u})\Delta y + \left( \frac{\partial f_0}{\partial y} \right)_{\bar{f}_0} \Delta y + \left( \frac{\partial f_0}{\partial u} \right)_{\bar{f}_0} \Delta u = \Delta \bar{f} \tag{3.10} \]

where

\[ \Delta \bar{f} = - \sum_{i=1}^{n} \bar{y}^{(i)} \left[ \left( \frac{\partial f_i}{\partial y} \right)_{\bar{f}_i} \Delta y + \left( \frac{\partial f_i}{\partial u} \right)_{\bar{f}_i} \Delta u \right] \tag{3.11} \]

**Note** that \( \Delta \bar{f} = 0 \) if the operating point is a steady-state with all \( \bar{y}^{(i)} = 0 \).
3.3 Model linearization

Constitutive relationships

Nonlinear constitutive relationships also need to be linearized. Such a relationship can be formally written

\[ g(z, y, u) = 0 \] (3.12)

where \( z \) is a new variable that is related to \( y \) and \( u \) according to (3.12). Linearization using a first-order Taylor series expansion as in (3.6) gives

\[
\left( \frac{\partial g}{\partial z} \right) \Delta z + \left( \frac{\partial g}{\partial y} \right) \Delta y + \left( \frac{\partial g}{\partial u} \right) \Delta u = 0
\] (3.13)

If the nominal operating point is a steady-state with all time derivatives zero, differentiation with respect to time gives for the \( i \):th time derivative

\[
\left( \frac{\partial g}{\partial z} \right) \Delta z^{(i)} + \left( \frac{\partial g}{\partial y} \right) \Delta y^{(i)} + \left( \frac{\partial g}{\partial u} \right) \Delta u^{(i)} = 0
\] (3.14)

If desired, the variable \( \Delta z \) can be introduced as dependent variable instead of \( \Delta y \) in (3.8) or (3.10) by means of (3.13) and (3.14).
Example 3.9.  Linearization of a first-order DE.

In Example 3.5 we derived the nonlinear DE

\[ A\dot{h} + a\sqrt{2gh} - u = 0 \]

We want to linearize this DE at the steady-state operating point \((\bar{h}, \bar{u})\).

Application of (3.10) gives

\[
A\Delta\dot{h} + \left( \frac{\partial}{\partial h} (a\sqrt{2gh} - u) \right)_{\bar{h}, \bar{u}} \Delta h + \left( \frac{\partial}{\partial u} (a\sqrt{2gh} - u) \right)_{\bar{h}, \bar{u}} \Delta u = 0
\]

\[
A\Delta\dot{h} + a\sqrt{2g} \left( \frac{d}{dh} \right)_{\bar{h}} \Delta h - \left( \frac{d}{du} \right)_{\bar{u}} \Delta u = 0
\]

which gives

\[
A\Delta\dot{h} + \frac{a\sqrt{2g}}{2\sqrt{\bar{h}}} \Delta h = \Delta u
\]
3.3 Model linearization

**Exercise 3.1.** Linearization of constitutive relationship.

Consider a control valve in a pipeline, schematically illustrated in the figure. At a given pressure, the flow rate $q$ through the valve depends on the position $x$ of the valve plug (stem) according to the valve characteristic

$$q = C(\alpha^x - 1)/(\alpha - 1)$$

where $C$ and $\alpha$ are constant parameters depending on the size and construction of the valve. The valve is closed when $x = 0$ and fully open when $x = 1$.

The position $x$ is adjusted by an actuator responding to a control signal $u$. Because of inertia, $x$ follows $u$ according to the dynamic relationship

$$T\ddot{x} + x = Ku$$

where $T$ and $K$ are constant parameters (time constant and static gain).

**Derive a linear dynamic model** that shows how $q$ depends on $u$ close to an operating point $q = \bar{q}$. 