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3. Mathematical Modelling

3.1 Modelling principles

3.1.1 Model types

For *design* and *analysis* of a control system we need a *mathematical model* that describes the dynamical behaviour of the system. The dynamics can be described by

- *differential equations* for *continuous-time* dynamics
- *difference equations* for *discrete-time* dynamics

Most processes are time continuous, but

- some processes are inherently time discrete (e.g. radioactive decay)
- *computer algorithms* (i.e. controllers) and many measuring devices produce outputs at *discrete time* instants
  - to design such controllers, we sometimes use discrete-time models to describe continuous-time processes

In this course, we will consider

- both types of models
- controller design both in continuous and discrete time

However, the major part of the course deals with continuous time.
3. Mathematical Modelling

3.1 Modelling principles

3.1.2 Model construction

There are two main principles for construction of mathematical models:

- **Modelling from first principles**: we derive models using physical laws and other known relationships (models).
- **System identification**: we use observations (measurements) of the system to find a model empirically. Usually, designed identification experiments are carried out to generate suitable data.

Often both methods are combined: we derive the basic model from first principles and determine uncertain parameters by system identification.

It is important to realize that *all models have a limited validity range*, even the physical laws (e.g. Newton’s laws of motion do not apply close to the speed of light). It is especially important to note that

- models determined through system identification should not be used outside the experimental range.
3.1.3 Modelling from first principles

In the following we consider modelling from first principles. Because real technical systems tend to be complex we cannot or do not want to include all details of the system in the model.

We try to make a **good compromise** between the following two requirements. The model should be

- **sufficiently accurate** for its intended purpose
- **simple enough** to use e.g. for system analysis and control design

In modelling from first principles, two types of mathematical relationships are used:

- *conservation laws*
- *constitutive relationships*
3.1 Modelling principles

Conservation laws

Conservation laws apply to additive quantities of the same type in a system. There are two general kinds of conservation laws:

- flow balances
- ”effort” balances

A flow balance for a given quantity in a system has the general form

\[
\frac{\text{accumulation}}{\text{time unit}} = \text{inflow} - \text{outflow} + \frac{\text{production}}{\text{time unit}}
\]

where

- accumulation and production occur inside the system
- inflow and outflow cross the system boundaries

Flow balances apply to conserved quantities (under normal conditions). If no chemical or nuclear reactions take place, the production is zero.

Examples of flow balance quantities:

- mass
- particles (moles)
- energy
- current (Kirchhoff’s first law)

Note that volume is not a conserved quantity for compressive fluids.
3.1 Modelling principles

An **effort balance** for a given quantity has the general form

\[
\frac{\text{change}}{\text{time unit}} = \text{forcing quantity} - \text{loading quantity}
\]

where

- change refers to a system property
- driving and loading refer to interaction with the surrounding

Generally, effort balances are applications of *Newton’s laws of motion* and *Kirchhoff’s second law*.

**Examples** of effort balance quantities:

- force
- momentum
- angular momentum
- voltage (*Kirchhoff’s second law*)
Constitutive relationships

Constitutive relationships are *static relationships* that relate quantities of different kinds in a system.

Examples of constitutive relationships:

- **Ohm’s law**: relates the current to the voltage over a resistance
- **valve characteristics**: relates the flow rate to the pressure drop over a valve
- **Bernoulli’s law**: relates the velocity of the flow out of a tank to the liquid level in the tank
- **the ideal gas law**: relates the temperature to the pressure of a gas in a closed container
3.1 Modelling principles

The general modelling procedure

1. Formulate balance equations.

2. Introduce constitutive relationships to
   – relate variables to each other;
   – possibly to introduce new variables in the balance equations.

3. Do a correctness check by at least checking that
   – all additive terms in an equation have the same physical unit;
   – the left and right hand side of an equation have the same unit.
3. Mathematical Modelling

3.2 Models for technical systems

3.2.1 Electrical systems

Fig. 3.1 shows three basic components of electrical circuits.

Variables
- \( t \) = time
- \( u \) = voltage [V]
- \( i \) = current [A]

Component parameters
- \( R \) = resistance [Ω]
- \( C \) = capacitance [F]
- \( L \) = inductance [H]

Relationships
- Resistor (Ohm’s law):
  \[ u(t) = R i(t) \] (3.1)
- Capacitor:
  \[ u(t) = u(0) + \frac{1}{C} \int_0^t i(\tau) \, d\tau \] (3.2)
- Inductor:
  \[ u(t) = L \frac{di(t)}{dt} \] (3.3)

Fig. 3.1. Basic components in electrical circuits.
**Example 3.1. A passive analog low-pass filter.**

Figure 3.2 shows a *passive analog low-pass filter.*

How does the voltage $u_{\text{out}}(t)$ on the output side depend on the voltage $u_{\text{in}}(t)$ on the input side if the circuit is uncharged at the output?

**Notation:**
- $u_R(t) = \text{voltage across the resistor, } i_R(t) = \text{current through the resistor}$
- $u_C(t) = \text{voltage across the capacitor, } i_C(t) = \text{current through the capacitor}$

According to *Kirchhoff’s second law*, the voltages satisfy

$$u_{\text{in}}(t) = u_R(t) + u_C(t) \quad (1)$$

$$u_{\text{out}}(t) = u_C(t) \quad (2)$$

When the output is uncharged, there is no current from the filter. Thus,

$$i_R(t) = i_C(t) \quad (3)$$
3.2.1 Electrical systems

Combination of (1) and (2) and elimination of $u_R(t)$ by (3.1) give

$$u_{out}(t) = u_{in}(t) - Ri_R$$

(4)

Elimination of $u_C(t)$ from (2) by (3.2) yields

$$u_{out}(t) = u_C(t) = u_C(0) + \frac{1}{C} \int_0^t i_C(\tau) \, d\tau$$

(5)

The derivative of both sides of (5) with respect to time gives

$$\frac{d u_{out}(t)}{dt} = \frac{1}{C} i_C(t) = \frac{1}{C} i_R(t)$$

(6)

where the last equality is given by (3). Combination of (4) and (6) to eliminate $i_R(t)$ gives

$$RC \frac{d u_{out}(t)}{dt} + u_{out}(t) = u_{in}(t)$$

(7)

This is a **linear first-order differential equation**. The circuit is a *low-pass filter* that filters (i.e., reduces the amplitude of) of high-frequency oscillations in $u_{in}(t)$.

In practical applications, the output is charged, which violates the assumption of this derivation. However, when an amplifier is used on the output side, (3) still holds (approximatively).
Example 3.2. A simple RLC circuit.

Figure 3.3 shows a simple RLC circuit charged by a current source.

How does the voltage across the capacitor depend on the current from the current source?

Notation:
- \( u_R(t) \) = voltage across the resistor, \( i_R(t) \) = current through the resistor
- \( u_C(t) \) = voltage across the capacitor, \( i_C(t) \) = current through the capacitor
- \( u_L(t) \) = voltage across the inductor, \( i_L(t) \) = current through the inductor

Kirchhoff’s laws give

\[
\begin{align*}
   u_C(t) &= u_R(t) + u_L(t) \\
   i(t) &= i_R(t) + i_C(t) \\
   i_R(t) &= i_L(t)
\end{align*}
\]
3.2.1 Electrical systems

Substitution of (3.1) and (3.3) into (1):

\[ u_C(t) = R i_R(t) + L \frac{di_L(t)}{dt} \]  (4)

Elimination of \( i_R(t) \) and \( i_L(t) \) by (2) and (3):

\[ u_C(t) = R (i(t) - i_C(t)) + L \frac{d(i(t) - i_C(t))}{dt} \]  (5)

According to eq. (6) in Ex. 3.1:

\[ i_C(t) = C \frac{du_C(t)}{dt} \]  (6)

Substitution of (6) into (5):

\[ u_C(t) = R \left( i(t) - C \frac{du_C(t)}{dt} \right) + L \frac{d(i(t) - C \frac{du_C(t)}{dt})}{dt} \]  (7)

After rearrangement:

\[ LC \frac{d^2u_C(t)}{dt^2} + RC \frac{du_C(t)}{dt} + u_C(t) = Ri(t) + L \frac{di(t)}{dt} \]  (8)

where \( i(t) \) is the input signal and \( u_C(t) \) is the output signal.

This is a **linear second-order differential equation**.
3.2.2 Mechanical systems

The modelling of mechanical systems are mainly based on Newton’s second law

\[ F = ma \] (3.4)

\( F \) is the force acting on the mass \( m \) and \( a \) is the acceleration of the mass.

Example 3.3. An undamped pendulum.

Figure 3.4 shows an undamped swinging pendulum. The pendulum can only move in two directions in the plane of the figure. Its point of suspension is at a distance \( u \) and its center of mass (the round weight at the lower end of the pendulum) is at a distance \( y \) from the left-side vertical line.

How does the position \( y \) depend on \( u \) ?

Notation:

- \( \ell \) = length of pendulum, \( m \) = weight of mass
- \( h \) = vertical position of the center of mass
- \( \theta \) = angle of swing away from a vertical position
- \( F \) = force acting on the suspension point in the “negative direction” (upwards)

![Fig. 3.4. Swinging pendulum.](image-url)
When the pendulum is affected by the suspension force \( F \) and the gravitational force \( mg \), *Newton’s second law* yields

- horizontal force components: \( m\ddot{y} = -F \sin \theta \)  

- vertical force components: \( m\ddot{h} = -F \cos \theta + mg \)

Here \( \ddot{y} \) and \( \ddot{h} \) are *second-order time derivatives* of \( y \) and \( h \), respectively, i.e. the *acceleration* in the respective directions.

Assume that the swing of the pendulum is moderate so that the *angle* \( \theta \) is always small. The pendulum then moves very little in the vertical direction and we can assume that \( \ddot{h} \approx 0 \). Elimination of \( F \) then gives

\[
\ddot{y} + g \tan \theta = 0
\]

The angle \( \theta \) is given by the trigonometric identity

\[
\tan \theta = \frac{y-u}{h} \approx \frac{y-u}{l}
\]

Combination of (3) and (4) yields the model

\[
\ddot{y} + \left( \frac{g}{l} \right) y = \left( \frac{g}{l} \right) u
\]

Notice that the approximations \( \ddot{h} \approx 0 \) and “\( \theta \) small” *limit the validity of the model*. 

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**3.2 Models for technical systems**

**3.2.2 Mechanical systems**

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**Process Dynamics and Control**
Example 3.4. Suspension system in a car.

Fig. 3.5. a) Spring-mounted mass with damping
b) Principle of car suspension system.
3.2.2 Mechanical systems

Example 3.4. Suspension system in a car

a) How does the vertical deviation \( y(t) \) from an equilibrium point depend on a force \( u(t) \) acting on the spring-mounted mass \( m \)?

An equilibrium point applies when \( y = u = 0 \) (units omitted). If the downward direction is the positive vertical direction, Newton’s second law for the spring force and the damping force of the cylinder gives

\[
m\ddot{y} = -b\dot{y} - ky + u(t) , \text{ i.e., } m\ddot{y} + b\dot{y} + ky = u(t)
\]  

(1)

where \( b \) and \( k \) are constants. The gravitational force \( mg \) is cancelled out when deviations from the equilibrium point are considered.

b) How do the deviations \( y_1(t) \) and \( y_2(t) \) in a car suspension system depend on \( u(t) \), which denotes the roughness of the ground?

\( m_1 \) is the mass of the car, \( m_2 \) is the mass of the wheels and the axles, \( b_1 \) and \( k_1 \) describe the dynamics of the car shock absorber, \( k_2 \) denotes the elasticity of the tires. In equilibrium, \( y_1 = y_2 = u = 0 \). The model becomes

\[
m_1\ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) = 0 \tag{2}
\]

\[
m_2\ddot{y}_2 - b_1(\dot{y}_1 - \dot{y}_2) - k_1(y_1 - y_2) = k_2(u - y_2) \tag{3}
\]

These are two coupled 2nd order differential equations that describe the vertical position of the car body and the wheels as function of the vertical roughness of the road.
3. Mathematical Modelling

3.2 Models for technical systems

3.2.3 Process engineering systems

Process engineering systems are typically modeled with flow balances (mass and energy balances) and constitutive relationships.

Example 3.5. Liquid tank with free outflow.

A volumetric flow rate $u$ is supplied continuously to a container and a volumetric flow rate $q$ flows out freely by gravity as determined by the height $h$ of the liquid the container.

The container has a constant cross-sectional area $A$, and the outlet tube has the “effective” cross-sectional area $a$. The liquid has a constant density $\rho$.

How does the level of the liquid depend on the inflow $u$?

Mass balance:

$$\frac{d}{dt} (\rho Ah) = \rho u - \rho q$$

(1)

Because $\rho$ and $A$ are constants, (1) can be simplified to

$$A \frac{dh}{dt} = u - q$$

(2)
According to *Bernoulli’s law*, the following *constitutive relationship* applies for the outflow of a liquid:

\[ v = \sqrt{2gh} \quad (3) \]

Here \( v \) is the velocity of the outflow and \( g \) is the acceleration due to gravity. The volumetric flow rate \( q \) is then given by

\[ q = av = a\sqrt{2gh} \quad (4) \]

where \( a \) is the effective cross-sectional area of the outflow tube, which is slightly smaller than the actual cross-sectional area \( A \). Combination of (2) and (4) now yields

\[ \frac{dh}{dt} + \frac{a\sqrt{2g}}{A} \sqrt{h} = \frac{1}{A} u \quad (5) \]

This is a **nonlinear differential equation** that describes how the level \( h \) depends on the inflow \( u \).
Example 3.6. A mixing tank.

Two inflows with the volumetric flow rates $F_1$ and $F_2$, and concentrations (mass/volume) $c_1$ and $c_2$ of some component $X$, are mixed continuously in a container. The outflow has the volumetric flow rate $F_3$ and concentration $c_3$. The container has a constant cross-sectional area $A$ and the height of liquid is $h$. The mixing in the container is assumed to be perfect so that the concentration of $X$ is $c$ everywhere in the container.

How do the level $h$ and the concentration $c$ depend on other variables?

If the two inflows have the same constant temperatures and “small” (or not too different) concentrations of $X$, it is reasonable to assume that the liquid density in the different flows is constant and the same.
As in Ex. 3.5, the density can be cancelled out yielding a

**total mass balance:** \[ A \frac{dh}{dt} = F_1 + F_2 - F_3 \] (1)

Since it is not specified what \( F_3 \) depends on, it cannot be eliminated now.

A mass balance can also be formed for each component \( X \):

**partial mass balance:** \[ \frac{d}{dt} (Ahc) = F_1 c_1 + F_2 c_2 - F_3 c_3 \] (2)

Because *perfect mixing* is assumed, the concentration is the same every-where in the container at a given time instant. This means that the

**constitutive relationship:** \[ c_3 = c \] (3)

must hold. Development of the derivative in (2) according to the product rule and elimination of \( c_3 \) by (3) yield

\[ Ac \frac{dh}{dt} + Ah \frac{dc}{dt} = F_1 c_1 + F_2 c_2 - F_3 c \] (4)

Combination of (4) with (1) gives

\[ Ah \frac{dc}{dt} = F_1 (c_1 - c) + F_2 (c_2 - c) \] (5)

This is a **linear differential equation** with (in general) **non-constant parameters** \( F_1, F_2, c_1, c_2 \).
Example 3.7. A water heater.

The inflow of water to the water heater has the mass flow rate $\dot{m}_1$ and temperature $T_1$ whereas the outflow has the mass flow rate $\dot{m}_2$ and temperature $T_2$. The mass of water in the heater is $M$ and it is heated to a temperature $T$ with a heating power $\dot{Q}$. The mixing of water in the heater is assumed to be perfect.

How do the amount of water and the temperature in the heater depend on other variables?

**Mass balance:**

$$\frac{dM}{dt} = \dot{m}_1 - \dot{m}_2$$

**Energy balance:**

$$\frac{dE}{dt} = \dot{E}_1 - \dot{E}_2 + \dot{Q}$$

Here, $\dot{E}_1$ and $\dot{E}_2$ are energy flows associated with the inflow and the outflow, respectively.
The energy in a substance is proportional to its mass or mass flow rate. For liquids it applies with good accuracy that the energy is also proportional to its temperature. This results in the

constitutive relationships: \[ E = c_p TM, \quad \dot{E}_1 = c_p T_1 \dot{m}_1, \quad \dot{E}_2 = c_p T_2 \dot{m}_2 \] (3)

Here \( c_p \) is the specific heat capacity for water, which in this case is assumed to be constant independently of the water temperature. Combination of (2) and (3) and development of the derivative according to the product rule give

\[ T \frac{dM}{dt} + M \frac{dT}{dt} = T_1 \dot{m}_1 - T_2 \dot{m}_2 + \frac{\dot{Q}}{c_p} \] (4)

Because of the assumption of perfect mixing, there is also a

constitutive relationship: \[ T_2 = T \] (5)

Elimination of \( dM/dt \) from (4) by (1) and substitution of (5) give

\[ M \frac{dT}{dt} = \dot{m}_1 (T_1 - T) + \frac{\dot{Q}}{c_p} \] (6)

Equation (1) and (6) show how the mass and the temperature in the heater depend on the inflow and the heating power \( \dot{Q} \).
3.2.3 Process engineering systems

If we want to use **volumetric units** instead of mass units in the model, this can easily be accomplished by the substitutions

\[ M = \rho Ah, \quad \dot{m}_1 = \rho_1 F_1 \]  

which applied to (6) yield

\[ \rho Ah \frac{dT}{dt} = \rho_1 F_1 (T_1 - T) + \frac{\dot{Q}}{c_p} \]  

Note that the **water density is not assumed to be constant** in equation (8).

Equation (1) expressed in volumetric units becomes more complicated when the water density is non-constant., i.e.,

\[ A \frac{d\rho h}{dt} = \rho_1 F_1 - \rho_2 F_2 = \rho_1 F_1 - \rho F_2 \]  

It is possible to show that even if \( \rho \neq \rho_1 \) due to the fact that \( T \neq T_1 \), the effects tend to cancel out in such a way that

\[ A \frac{dh}{dt} \approx F_1 - F_2 \]  

becomes a good approximation of (1) and (9).
Example 3.8. Gas flow through a tank.

Figure 3.9 illustrates a closed gas tank with the volume $V$, amount of substance $n$ (number of moles), pressure $p$, and temperature $T$. The inflow to the tank is the molar flow rate $\dot{n}_1$ at pressure $p_1$ before Valve 1, the outflow is the molar flow rate $\dot{n}_2$ at pressure $p_2$ after Valve 2. Valve 2 can be used for control by adjustment of the valve position $u$.

How does the pressure $p$ in the tank depend on other variables?

**Molar balance:**

\[
\frac{dn}{dt} = \dot{n}_1 - \dot{n}_2 \tag{1}
\]

The molar flow rate through a valve with a given opening position can be assumed to be proportional to the square root of the pressure difference across the valve. It is also reasonable to assume that the proportionality factor is proportional to the square of the “linear” valve position $u$. Thus:

**Constitutive relationships:**

\[
\dot{n}_1 = k_1 \sqrt{p_1 - p} \quad \dot{n}_2 = k_2 u^2 \sqrt{p - p_2} \tag{2}
\]
3.2.3 Process engineering systems

In the absence of other information, we can assume that the ideal gas law holds. Thus, we have

**Constitutive relationship:**

\[ pV = nRT \]  

(3)

Here, \( R \) is the general gas constant and \( T \) is the temperature expressed in Kelvin.

If the temperature \( T \) is constant, substitution of (2) and (3) in (1) gives

\[ \frac{dp}{dt} = \frac{RT}{V} \frac{dn}{dt} = \frac{RT}{V} \left( k_1 \sqrt{p_1 - p} - k_2 u^2 \sqrt{p - p_2} \right) \]  

(4)

which is a relatively complex nonlinear differential equation, even if it is of first order.
3. Mathematical Modelling

3.3 Model linearization

3.3.1 Motivation

We have in a number of examples shown how to derive dynamic models for many types of technical systems. In all cases, the obtained models are ordinary differential equations. We note that

- the differential equations (DEs) are often nonlinear
- even if they are linear, the coefficients are generally not constant because they depend on some physical time-varying quantity
- it is difficult, maybe impossible, to find general solutions to these kinds of DEs

Therefore, we need to

- study special cases and/or
- do simplifying assumptions

Frequently used simplifications is to assume that

- some quantities are constant, even if they are (slightly) time-varying
- input signals change in some ideal (but reasonable) way
3.3  Model linearization

In practice, it is often enough to know the system behaviour in a limited region close to a known operating point. Then, the model simplification may be to

- linearize the model equations at the operating point.

The advantage of this is that

- efficient analysis, synthesis, and design methods based on linear algebra can be used.

If the system is very nonlinear, or the operating region very large, one can use

- several linear models that are linearized at different operating points.

Because of the reasons mentioned above,

- modelling from first principles is often followed by a linearization of the model.

In this course, we are only considering models obtained from ordinary differential equations, not partial DEs.
3.3.2 Linearization of ODEs

A general ODE

Consider an \( n \):th order ODE, which we can formally write as

\[
f(y^{(n)}, \ldots, \dot{y}, y, u) = 0anumber{3.5}
\]

- For simplify, time derivatives of \( u \) are not included; they can be handled in the same way as the time derivatives of \( y \).
- Usually the time derivatives appear linearly in (3.5); however, the linearization applies also when they do not appear linearly.

Equation (3.5) can be linearized by a \textit{first-order Taylor series expansion} at the \textit{nominal operating point} \((\bar{y}^{(n)}, \ldots, \bar{y}, \bar{y}, \bar{u})\), denoted by \(\bar{f}\):

\[
f(y^{(n)}, \ldots, \dot{y}, y, u) \equiv f(\bar{y}^{(n)}, \ldots, \bar{\dot{y}}, \bar{y}, \bar{u}) + \left(\frac{\partial f}{\partial y^{(n)}}\right)_{\bar{f}} (y^{(n)} - \bar{y}^{(n)}) + \\
\ldots + \left(\frac{\partial f}{\partial \dot{y}}\right)_{\bar{f}} (\dot{y} - \bar{\dot{y}}) + \left(\frac{\partial f}{\partial y}\right)_{\bar{f}} (y - \bar{y}) + \left(\frac{\partial f}{\partial u}\right)_{\bar{f}} (u - \bar{u})anumber{3.6}
\]

- Usually the operating point is a \textit{static (steady-state) point} with all time derivatives equal to zero, but (3.6) holds even if this is not so.
We introduce the variables

\[
\Delta y^{(n)} \equiv y^{(n)} - \bar{y}^{(n)}, \ldots, \Delta \dot{y} \equiv \dot{y} - \bar{y}, \Delta y \equiv y - \bar{y}, \Delta u \equiv u - \bar{u}
\]  

(3.7)

which denote deviations from the nominal operating point. We call such variables *deviation variables*, or simply, *Δ-variables*.

Combination of (3.5), (3.6) and (3.7) with the fact that the nominal operating point has to satisfy (3.5) yields

\[
\left( \frac{\partial f}{\partial y^{(n)}} \right)_\bar{f} \Delta y^{(n)} + \ldots + \left( \frac{\partial f}{\partial \dot{y}} \right)_\bar{f} \Delta \dot{y} + \left( \frac{\partial f}{\partial y} \right)_\bar{f} \Delta y + \left( \frac{\partial f}{\partial u} \right)_\bar{f} \Delta u = 0
\]  

(3.8)

This is a *linear n:th order ODE with constant coefficients*.

**Note:** If the ODE contains time derivatives of \( u \), they appear as Δ-variables in (3.8) in the same way as the time derivatives of \( y \) appear.
3.3 Model linearization

**ODEs linear in the time derivatives**

If the time derivatives appear linearly in (3.5), we can formally write

\[ f_n(y, u)y^{(n)} + \cdots + f_1(y, u)y' + f_0(y, u) = 0 \]  

(3.9)

Equation (3.6) can now be used to linearize every term separately giving

\[ f_0(y, u) \approx f_0(\bar{y}, \bar{u}) + \left(\frac{\partial f_0}{\partial y}\right)\bar{f}_0 \Delta y + \left(\frac{\partial f_0}{\partial u}\right)\bar{f}_0 \Delta u \]

The term with the \(i\):th derivative results in

\[ f_i(y, u)y^{(i)} \approx f_i(\bar{y}, \bar{u})\bar{y}^{(i)} + f_i(\bar{y}, \bar{u})\Delta y^{(i)} + \left(\frac{\partial f_i}{\partial y}\right)\bar{f}_i \Delta y + \left(\frac{\partial f_i}{\partial u}\right)\bar{f}_i \bar{y}^{(i)} \Delta u \]

Substitution into (3.9) and cancelling out constant terms give

\[ f_n(\bar{y}, \bar{u})\Delta y^{(n)} + \cdots + f_1(\bar{y}, \bar{u})\Delta y + \left(\frac{\partial f_0}{\partial y}\right)\bar{f}_0 \Delta y + \left(\frac{\partial f_0}{\partial u}\right)\bar{f}_0 \Delta u = \Delta \bar{f} \]  

(3.10)

where

\[ \Delta \bar{f} = -\sum_{i=1}^{n} \left(\bar{y}^{(i)} \left[ \left(\frac{\partial f_i}{\partial y}\right)\bar{f}_i \Delta y + \left(\frac{\partial f_i}{\partial u}\right)\bar{f}_i \Delta u \right] \right) \]

(3.11)

**Note** that \(\Delta \bar{f} = 0\) if the operating point is a steady-state with all \(\bar{y}^{(i)} = 0\).
Constitutive relationships

Nonlinear constitutive relationships also need to be linearized. Such a relationship can be formally written

\[ g(z, y, u) = 0 \]  \hspace{1cm} (3.12)

where \( z \) is a new variable that is related to \( y \) and \( u \) according to (3.12). Linearization using a first-order Taylor series expansion as in (3.6) gives

\[ \frac{\partial g}{\partial z} \Delta z + \frac{\partial g}{\partial y} \Delta y + \frac{\partial g}{\partial u} \Delta u = 0 \]  \hspace{1cm} (3.13)

If the nominal operating point is a steady state with all time derivatives zero, differentiation with respect to time gives for the \( i \):th time derivative

\[ \left( \frac{\partial g}{\partial z} \right) \Delta z^{(i)} + \left( \frac{\partial g}{\partial y} \right) \Delta y^{(i)} + \left( \frac{\partial g}{\partial u} \right) \Delta u^{(i)} = 0 \]  \hspace{1cm} (3.14)

If desired, the variable \( \Delta z \) can be introduced as dependent variable instead of \( \Delta y \) in (3.8) or (3.10) by means of (3.13) and (3.14).
### 3.3 Model linearization

#### 3.3.2 Linearization of ODEs

**Example 3.9. Linearization of a first-order DE.**

In Example 3.5 we derived the nonlinear DE

\[ A\dot{h} + a\sqrt{2gh} - u = 0 \]  

(1)

We want to linearize this DE at the steady-state operating point \((\bar{h}, \bar{u})\).

Application of (3.10) gives

\[ A\Delta \dot{h} + \left( \frac{\partial}{\partial h} (a\sqrt{2gh} - u) \right)_{\bar{h}, \bar{u}} \Delta h + \left( \frac{\partial}{\partial u} (a\sqrt{2gh} - u) \right)_{\bar{h}, \bar{u}} \Delta u = 0 \]

\[ \Rightarrow A\Delta \dot{h} + a\sqrt{2g} \left( \frac{d\sqrt{h}}{dh} \right)_{\bar{h}} \Delta h - \left( \frac{du}{du} \right)_{\bar{u}} \Delta u = 0 \]

(2)

which gives

\[ A\Delta \dot{h} + \frac{a\sqrt{2g}}{2\sqrt{h}} \Delta h = \Delta u \]

(3)
Exercise 3.1. Linearization of constitutive relationship.

Consider a control valve in a pipeline, schematically illustrated in the figure. At a given pressure, the flow rate $q$ through the valve depends on the position $x$ of the valve plug (stem) according to the valve characteristic

$$q = C(\alpha^x - 1)/(\alpha - 1)$$

where $C$ and $\alpha$ are constant parameters depending on the size and construction of the valve. The valve is closed when $x = 0$ and fully open when $x = 1$.

The position $x$ is adjusted by an actuator responding to a control signal $u$. Because of inertia, $x$ follows $u$ according to the dynamic relationship

$$T \dot{x} + x = Ku$$

where $T$ and $K$ are constant parameters (time constant and static gain).

Derive a linear dynamic model that shows how $q$ depends on $u$ close to an operating point $q = \bar{q}$. 