A system is a combination of components that act together to perform a certain objective. A signal is (usually) a physical quantity or variable. Depending on the context, the term "signal" may refer to the type of variable (e.g., a temperature) or the value of a variable (e.g., a temperature expressed as a numerical value). In practice, this does not cause confusion.

There are two types of input signals:
- Control signals are inputs whose values we can adjust.
- Disturbances are inputs whose values we cannot affect (in a rational way).

Generally, signals are functions of time, which can be indicated by $u(t)$ and $y(t)$.

A system is static if the outputs are completely determined by the inputs at the same time instant; such behavior can be described by algebraic equations. A system is dynamic if the outputs depend also on inputs at previous time instants; such behavior can be described by differential equations.

In practice, this does not cause confusion.

There are two main types of signals:
- Output signals are physically measured or calculated.
- Input signals are physically measured or calculated.

A system interacts with its environment through signals. In particular, the value of a signal may be known if it is a measured variable.

In practice, this does not cause confusion.

The value of a signal may be known if it is a measured variable. In particular, the value of a variable may be known if it is a measured variable. Depending on the context,
The flow through the control valve depends on the valve position, \( n \), and the primary pressure, \( p' \), and the secondary pressure, \( p'' \). The flow is the output signal of the system, whereas \( x \) and \( y \) are the input signals.

The flow \( b \) depends on the other variables in a dynamic way. The relationship between the steady-state values of the variables is relatively slow. The valve characteristics are provided by the valve manufacturer (given a specific type of control valve, \( A \)).

The blocks in a block diagram consisting of several blocks are connected via their signals. The simplest form a block diagram is a single block, illustrated by Fig. 2.1. The interior of a block usually contains a description or the name of the corresponding system, or a symbol for the mathematical operation on the input to yield the output. The signals are represented by arrows, which show the direction of information flow. The blocks in a block diagram consist of several blocks and are connected via their signals. A block diagram is a pictorial representation of cause-and-effect relationships between signals.
2.4 Control Strategies

The input and output signals in control systems are not necessarily streams in a literal sense, and even if they are, the signal direction does not have to be the same as the direction of the corresponding physical stream.

For instance, a physical outlet stream may well be a control input signal as shown in Ex. 2.2 on next slide.

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If $v = 1 \text{ m/s}$, then $h = 60 \text{ m}$. The temperature of the heated liquid is controlled by the flow rate of steam.

Examples of open-loop control

In some simple applications, open-loop control without measurements can be used. In this control strategy, the controller is tuned using some a priori information about the process. After the tuning has been made, the control action is a function of the setpoint only (setpoint = desired value of the controlled variable).

This control strategy has some advantages, but also clear disadvantages.

Examples of open-loop control

For instance, a physical outlet stream may well be a control input signal as shown in Ex. 2.2 on next slide.

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The block diagram is meant to illustrate the following:

- the controller is tuned using some a priori information about the process
- the controller is used to achieve some a priori information about the process
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This control strategy has some advantages, but also clear disadvantages.

Examples of open-loop control
Control is clearly needed to eliminate the effect of disturbances on the system output. Feedforward control is a type of open-loop control strategy, which can be used for disturbance elimination, if disturbances can be measured and it is known how the disturbances affect the output. It is known how the control signal affects the output. Feedforward is an open-loop control strategy because the output, which we want to control, is not measured. Obviously, this control strategy has advantages, but it also has some disadvantages. When feedforward control is used, it is usually used in combination with feedback control.

The figures below illustrate the heating of a house by (a) feedforward, (b) feedback control. Some advantages and disadvantages can be noted:

Feedforward: Rapid control because the controller acts before the effect of the disturbance (outdoor temperature) is seen in the output signal (indoor temperature), but requires good knowledge of the process model;

Feedforward: Rigid control because the controller acts before the effect of the disturbance (outdoor temperature) is seen in the output signal.

Feedback: Slower control because the controller does not act before the effect of the disturbance (outdoor temperature) is seen in the output signal; less sensitive to modelling errors and disturbances.

What would open-loop control of the indoor temperature look like?

Consider the two flow control diagrams below. Indicate the control strategies (feedback or feedforward) in each case and justify the answer. It can be assumed that the distance between the flow transmitter FT and the control valve is small.
The liquid tank to the right has an inflow and an outflow. The inflow is controlled so that $I = 10$ l/min. The volume of the liquid is desired to remain constant at $V = 1000$ liters. The inflow is measured and controlled so that $I = 10$ l/min. The liquid level is measured and controlled so that $h = 1000$ mm. These two types of control tasks may well be handled simultaneously. The task of the control system is to make the output signal follow the setpoint with as little error as possible. This is sometimes referred to as a tracking problem. The setpoint and the measured value of the output signal form the input signal to the controller. The controller is usually a constant and the output signal is the control deviation or control error. The output signal (at a certain instant) is sometimes called the actual value.

Two types of control can be distinguished depending on whether the setpoint is mostly constant or changes frequently:

- **Regulatory control.** The setpoint is usually constant and the main objective of the control system is to maintain the output signal at the setpoint, i.e., a regulatory problem.
- **Tracking control.** The setpoint varies and the main objective of the control system is to make the output signal follow the setpoint with as little error as possible, i.e., a servo problem.

These two types of control tasks may well be handled simultaneously; the differences arise in the choice of parameter values for the controller (Chapter 7).
If the inside temperature is below the desired temperature, the controller increases the heating power when the difference between the desired and actual inside temperatures is positive. The controller gain, denoted by $K$, is the ratio of the output change to the change in input (if $K > 0$).

$$K = \frac{\text{output change}}{\text{input change}}$$

The temperature inside the house depends on the outside temperature $T_o$ and the heating power $P$.

$$T = T_o + P$$

We can make the output signal stay at or follow a desired value $T_d$ by adjusting the controller gain $K$.

$$\text{output} = K \times \text{input}$$

In this example, the output signal is adjusted in order to produce a desired value. Normally, we do not need to know the characteristics of the system in order to tune the controller.

The temperature $T$ is the output (in this example on the controlled system). In order to avoid the effect of disturbances (the outside temperature $T_o$), it can be assumed that $T = T_o$ in the simplest case (the outside temperature known).

The output signal $y$ is equal to the desired temperature $T_d$.

In this case, $K > 0$, Equation (2.2) shows that the steady-state value is $y = T_d$.

We shall illustrate some fundamental properties of feedback control by looking at the relationship between the controller gain $K$ and the system gain $T/T_o$.

$$\frac{T}{T_o} = K$$

The system gain changes the output change for a given input change. The system gain is independent of the controller gain $K$.

$$\frac{T}{T_o} = \lim_{K \to \infty} \left( \frac{T}{T_o} \right)$$

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The system gain changes the output change for a given input change. The system gain is independent of the controller gain $K$.
Although we would desire $Q_2$ to be zero, we cannot achieve this in practice.

When $\alpha$ is the control valve and $\beta$ is the mixing point, the expression (2.10) shows that the best control with $Q_2$ is

$$Q_2 = \alpha \beta$$

The control valve is neglected.

If $\gamma > \alpha$, then $Q_2 = \alpha \beta$ is achieved.

If $\gamma < \alpha$, then $Q_2 = \alpha \beta$ is not achieved.

It is clear by the above example that we cannot achieve a controller in the mixing point.

In a simple way the disadvantages of heat at least in principle can be avoided by

In the example above we neglected the system dynamics in order to illustrate

Consider a steady state (2.8) is achieved.

Substituting equation (2.9) into equation (2.6) yields

expression then applies:

Consider a steady state (2.8) according to equation (2.6) the following

$$Q_2 = \alpha \beta$$

The general expression for flow becomes

According to equation (2.8), the result is almost the same as the

When the heat loss from the tank is neglected.

The mixing point reach the measuring point in practice.

Because the flow of water is $m^2/s$ this means that the temperature $\theta$ is

Moreover, the properties of the system to be controlled generally limit the

In addition, the control of the system to be controlled directly using the

Although we would desire $Q_2$ (2.9) to be zero, the normal steady state.

where $\alpha$ is the control valve and $\beta$ is a constant value of the mass flow

Then $Q_2$ (2.9) is used for control of $Q_2$ with the control valve is neglected.

Here is time expressed in minutes and $Q_2$ is positive process gain.

The expression then applies:

When the heat loss from the tank is neglected.

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The expression then applies:

When the heat loss from the tank is neglected.

The mixing point reaches the measuring point in practice.

Because the flow velocity $v$ is $m/s$ this means that the temperature $\theta$ is
Chapter 7.

In practice, there are many other modifications of the ideal PID controller. See also Figures 7.2 and 7.4. For example, an integral controller is obtained by letting \( K_i = 0 \) and

\[
\frac{\text{DP} + \text{LP} \cdot \text{LP} + \text{FP} \cdot (\text{LP} + \text{ID}) \cdot \text{ID}}{\text{ID} + \text{ID} + \text{ID}} \]

(2.12)

By losing some adjustable controller parameters, parts of the controller that are not needed can be disabled.

### 2.5 Feedforward control

#### 2.5.4 The PID controller

Now, for example, a pure integral controller is obtained by letting \( K_i = 0 \), and

\[
\frac{\text{DP} + \text{DP} \cdot \text{DP} + \text{FP} \cdot (\text{DP} + \text{ID}) \cdot \text{ID}}{\text{ID} + \text{ID} + \text{ID}} \]

(2.11)

A controller is obtained by \( K_i = \infty \), and \( K_p = 0 \). A derivative controller is obtained by letting \( K_i = 0 \). A so-called 

### 2.5.4 The PID controller

The so-called PID controller is defined by the differential of the controller are, in addition to the initial output, allowed at Exam Day.

The so-called PID controller is defined by the differential of the controller, i.e., the derivative term. The derivative term is given by

\[
\int_{t_0}^{t} \frac{d^2}{dt^2} \left( y(t) \right) dt = (1) \]

(2.10)

where \( y(t) \) is the control error, \( y(t) = (1) \) is the difference between the reference value and the measured value, i.e., the control error, and \( (1) \) is the control action.

By losing some adjustable controller parameters, parts of the controller that are not needed can be disabled.
The PI-controller is without doubt the most common controller in the (process) industry, where it is especially used for flow control. The PI-controller has good static properties, because it eliminates stationary control deviation, which reduces the stability (the integral collects old data!).

The D-effect is often included (PD or PID) in the control of processes with slow dynamics, especially temperature and vapour pressure. The D-effect gives good dynamic properties and good stability (the derivative "predicts" the future!); sensitivity to measurement noise.

Consider a PI-controller and assume that steady-state conditions apply for times \( t \geq t_0 \). This means that \( e(t) = 0 \) and \( e' = 0 \), respectively, for \( t \geq t_0 \).

Explain why this implies that \( e(t) = 0 \), i.e., that the control deviation must be zero at steady state.

Consider the double-integral controller (PII controller) \( e(t) = K_e \, e(t) + \frac{1}{T_i} \int_0^t e(\tau) \, d\tau + \frac{1}{T_d} \, e'(t) \), \( e(t) = K_e \, e(t) + \frac{1}{T_i} \int_0^t e(\tau) \, d\tau \).

What steady-state properties does this controller have? Explain why this means that \( e(t) = 0 \) and \( e' = 0 \) for steady-state conditions apply.

It is important to distinguish between negative feedback and positive feedback.

Negative feedback means that the control signal amplifies the control error.

Positive feedback means that the control signal reduces the control error.

What kind of feedback — positive or negative — should be used in a control system?

1. How do you know what kind of feedback you have in a control system?

2. What kind of feedback — positive or negative — should be used in a control system?

3. Which controller is without doubt the most common controller in the (process) industry?
For design and analysis of a control system we need a mathematical model that describes the dynamical behavior of the system. The dynamics can be described by differential equations for continuous-time dynamics and difference equations for discrete-time dynamics. Most processes are time continuous, but some processes are inherently time discrete (e.g., radioactive decay) or computer algorithms (i.e., controllers) and many measuring devices produce outputs at discrete time instants. To design such controllers, we sometimes use discrete-time models to describe continuous-time processes.

In this course, we will consider both types of models and controller design both in continuous and discrete time. However, the major part of the course deals with continuous time. It is important to realize that all models have a limited validity range, even the physical laws (e.g., Newton's laws of motion) do not apply close to the speed of light. It is especially important to note that models determined through system identification should not be used outside the experimental range.

There are two main principles for construction of mathematical models: modelling from first principles and system identification. Often both methods are combined: we derive the basic model from first principles and determine uncertain parameters by system identification. It is important to realize that all models have a limited validity range, even the physical laws (e.g., Newton's laws of motion) do not apply close to the speed of light. It is especially important to note that models determined through system identification should not be used outside the experimental range.

In the following, we consider modelling from first principles. Because real technical systems tend to be complex, we cannot or do not want to include all details of the system in the model. We try to make a good compromise between the following two requirements: the model should be sufficiently accurate for its intended purpose and simple enough to use for system analysis and control design. In modelling from first principles, two types of mathematical relationships are used:

- Conservation laws
- Constitutive relationships

Conservation laws apply to additive quantities of the same type in a system. There are two general kinds of conservation laws:

- Flow balances
- “Effort” balances

A flow balance for a given quantity in a system has the general form:

\[ \frac{d}{dt} \text{Accumulation} + \text{Production} = \text{Inflow} - \text{Outflow} \]

where Accumulation and Production occur inside the system, Inflow and Outflow cross the system boundaries, and where the net inflow and net outflow are zero. Conservation laws apply to additive quantities of the same type in a system. Conservation laws are used to describe the behavior of systems, and when combined with other laws and principles, we derive models using physical laws. There are two main principles for construction of mathematical models: modelling from first principles, and system identification.
Process Control Laboratory

An effort balance for a given quantity has the general form

\[
\frac{d}{dt} \mathbf{Y} = \mathbf{D} \mathbf{Y} + \mathbf{F}
\]

where

- \( \mathbf{D} \) change refers to a system property
- \( \mathbf{F} \) driving and loading refer to interaction with the surrounding environment

Generally, effort balances are applications of Newton's laws of motion and Kirchhoff's second law.

Examples of effort balance quantities:

- Force
- Momentum
- Angular Momentum
- Voltage (Kirchhoff's second law)

Constitutive relationships are static relationships that relate quantities of different kinds in a system.

Examples of constitutive relationships:

- Ohm's law: relates the current to the voltage over a resistance
- Valves characteristics: relates the flow rate to the pressure drop over a valve
- Bernoulli's law: relates the velocity of the flow out of a tank to the liquid level in the tank
- the ideal gas law: relates the temperature to the pressure of a gas in a closed container

Balanced level in the tank
- Boyle's law: relates the volume of a fixed mass of gas to the pressure upon it

Examples of constitutive relationships:

- Different kinds of systems:
  - Electric circuits
  - Fluid systems

The general modeling procedure:

1. Formulate balance equations.
2. Introduce constitutive relationships that relate variables to each other; possibly to introduce new variables in the balance equations.
3. Do a dimensional check by checking that all additive terms in an equation have the same units.
4. Check that the left and right hand side of an equation have the same units.
   - Additive terms in an equation have the same physical unit.
   - All dimensions check by checking that all dimensions are the same.

Examples of effort balance quantities that relate quantities of different kinds in a system:

- Force
- Momentum
- Angular Momentum
- Voltage (Kirchhoff's second law)
This is a linear second-order differential equation

\[
\frac{d^2}{dt^2} y(t) + 2 \frac{dy}{dt} + y(t) = 0
\]

where the input signal and \( y(t) \) is the output signal.

Substitution of \( \psi(t) = e^{\lambda t} \) into (5) gives

\[
\lambda^2 + 2\lambda + 1 = 0
\]

Solution:

\[
\lambda = -1, -1
\]

Elimination of \( \psi(t) \) from (1) and (2) yields

\[
\frac{d}{dt} y(t) + y(t) = 0
\]

Substitution of \( \psi(t) = e^{\lambda t} \) into (1) and (2)

Kirchhoff's laws: \( V = I R \) and \( E = I L \)

Circuit analysis:

1. Voltage across the resistor = \( V_R(t) = I(t) R \)
2. Current through the capacitor = \( C \frac{dV_C}{dt} = \psi(t) \)
3. Current through the inductor = \( L \frac{dI_L}{dt} = \psi(t) \)

Equation:

\[
\frac{d^2}{dt^2} y(t) + 2 \frac{dy}{dt} + y(t) = 0
\]

4. Various types of circuits:

- Circuit 3.2: Passive RLC circuit
- Circuit 3.3: Active RLC circuit

Experiment 3.1: A passive band-pass filter

According to Kirchhoff's second law, the voltage applied to the circuit is zero.

\[
\frac{d^2}{dt^2} y(t) + 2 \frac{dy}{dt} + y(t) = 0
\]

Combination of (1) and (2) yields

\[
\frac{d^2 y(t)}{dt^2} + 2 \frac{dy}{dt} + y(t) = 0
\]

Example 3.2: A simple RLC circuit

3.2.1 Electrical systems
Notice that the approximations that describe \( \kappa \)
are small. Hence, the validity of

\[
\frac{1}{n - \alpha} \approx \frac{1}{\alpha}
\]

combination of (3) and (4) yields the model

\[
\text{The angle } \theta \text{ is given by the trigonometric identity}
\]

\[
\sin \theta = \frac{\text{height of mass}}{\text{mass}}
\]

Elimination of \( n \) and \( \alpha \) gives the trigonometric identity

\[
\sin \theta = \frac{\text{height of mass}}{\text{mass}}
\]

A allowed at Exam – \( 4 \)

The model

\[
\text{The model}
\]

3.2 Mechanical systems

Example 3.4: Suspension system in a car

Example 3.4: Suspension system in a car

3.2 Mechanical systems

3.2 Models for mechanical systems

3. Mathematical Modelling

3.1 Process diagrams and control

3.1 Process diagrams and control

Figure 3.5: a) Spring-loaded mass with damper.

How does the vertical deviation \( \gamma \) from the suspension point in the

Figure 3.5: a) Spring-loaded mass with damper.

How does the vertical deviation \( \gamma \) from the suspension point in the

Process diagrams and control
This is a non-linear differential equation that describes how the level y in the container varies with time t. The differential equation is:

\[ \frac{dy}{dt} = \frac{\dot{V}}{A} \]

where \( A \) is the area of the outflow tube and \( \dot{V} \) is the volumetric flow rate.

Because perfect mixing is assumed, the concentration is the same everywhere in the container. This means that the concentration is a constant and can be eliminated now.

A mass balance can also be formed for each component. For component 1, the mass balance is:

\[ \frac{d}{dt} (\rho_1 \cdot V) = \frac{\dot{V}}{A} \cdot \rho_1 \]

where \( \rho_1 \) is the density of component 1.

The container has a constant cross-sectional area, so that the concentration of component 1 is the same everywhere in the container. The height of the liquid is the same as the height of the container.

The outflow has the volumetric flow rate \( \dot{V} \), and the cross-sectional area of the outflow tube is \( A \). The acceleration due to gravity is \( g \), and the velocity of the outflow is \( v \). The density of the liquid is \( \rho \).

The partial mass balance for component 1 is:

\[ \frac{d}{dt} (\rho_1 \cdot V) = \dot{V} \rho_1 \]

and the total mass balance is:

\[ \frac{d}{dt} (\rho_1 \cdot V) = \dot{V} \rho_1 + \dot{m}_{in} \rho_1 - \dot{m}_{out} \rho_1 \]

where \( \dot{m}_{in} \) and \( \dot{m}_{out} \) are the mass flow rates of the inflows and outflows, respectively.

The container is assumed to be a perfect mixer, so that the concentration of component 1 is the same everywhere in the container.

The height of the liquid is the same as the height of the container.

The outflow has the volumetric flow rate \( \dot{V} \), and the cross-sectional area of the outflow tube is \( A \). The acceleration due to gravity is \( g \), and the velocity of the outflow is \( v \). The density of the liquid is \( \rho \).
The energy in a substance is proportional to its mass or mass flow rate. The mixing of water in the heater is assumed to be isothermal, which results in the energy balance.

\[ \frac{d}{dt} \left( \rho V \right) + \left( \frac{d}{dx} \right) \left( \rho V \right) \left( \frac{d}{dx} \right) = \frac{d}{dt} \left( \rho \right) \frac{d}{dx} \left( V \right) \]

where \( \rho \) is the density of water and \( V \) is the volume of the tank.

The mixture of water in the heater is assumed to be isothermal, which is a good approximation of (1) and (9).
In the absence of other information, we can assume that the ideal gas law holds. Thus, we have

\[ \text{Constitutive relationship:} \]

\[ \rho = \rho_0 \left( \frac{T}{T_0} \right)^{3/2} \]

Here, \( \rho_0 \) is the general gas constant and \( T \) is the temperature expressed in Kelvin.

If the temperature \( T \) is constant, substitution of (2) and (3) in (1) gives

\[ \frac{\partial^2 \rho}{\partial \rho^2} \frac{\partial T}{\partial \rho} \frac{\partial T}{\partial \rho} \]

(4)

which is a cubic differential equation.

Therefore, we need to study special cases and/or do simplifying assumptions to make the algebra manageable.

In practice, it is often enough to know the system behaviour in a limited region close to a known operating point. Then, the model simplification may be to linearize the model equations at the operating point.

The advantage of this is that efficient analysis, synthesis, and design methods based on linear algebra can be used.

If the system is very nonlinear, or the operating region very large, one can use several linear models that are linearized at different operating points. Because they depend on some physical time-varying quantity, the model equations (3.5) are generally nonlinear. Therefore, we need to study special cases and/or do simplifying assumptions to make the algebra manageable.

Frequently used simplifications is to assume that some quantities are constant, even if they are (slightly) time-varying input signals change in some ideal (but reasonable) way.
If the operating point is a steady-state with all time derivatives zero, we can linearize the DE as follows:

\[ 0 = n \frac{\partial y}{\partial n} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} \]

Equation (3.10) can now be used to linearize every term separately giving

\[ N_{y} = n \frac{\partial y}{\partial n} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} \]

Note: If the ODE contains the time derivatives of \( y \) or \( \frac{\partial y}{\partial n} \), such variables need to be linearized. Also, note that the nonlinear constitutive relationships also need to be linearized. Such a relationship can be formally written as follows:

\[ N_{y} = n \frac{\partial y}{\partial n} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n} \frac{\partial^2 y}{\partial n^2} \]

where \( n \) is the new variable that is related to \( y \).
Consider a control valve in a pipeline, schematically illustrated in the figure. At a given pressure, the flow rate \( q \) through the valve depends on the position \( x \) of the valve plug (stem) according to the valve characteristic

\[
q = C(1 - e^{-\alpha x})
\]

where \( C \) and \( \alpha \) are constant parameters depending on the size and construction of the valve. The valve is closed when \( x = 0 \) and fully open when \( x = 1 \).

The position \( x \) is adjusted by an actuator responding to a control signal \( u \). Because of inertia, \( x \) follows \( u \) according to the dynamic relationship

\[
x' + \beta x = \gamma u
\]

where \( \beta \) and \( \gamma \) are constant parameters (time constant and static gain). Derive a linear dynamic model that shows how \( q \) depends on \( u \) close to an operating point \( q = q_0 \).

In practice, all real processes (systems) are nonlinear to some degree. However, there are several reasons why we want to study linear systems:

1. Linear systems are easier to model and analyze.
2. Linear models are often used as approximations of nonlinear systems.
3. Linear systems have well-defined properties and behaviors.
4. Linear control theory provides powerful tools for designing control systems.

A linear ODE has the general form

\[
\frac{d^m y}{dt^m} + \frac{d^{m-1} y}{dt^{m-1}} + \cdots + \frac{d y}{dt} + y = f(t)
\]

where \( y \) is the output from the system, \( f(t) \) is the input to the system. The order of the highest derivative of \( y \) is the system order \( n \). A system is proper if \( n \leq m \), it is strictly proper if \( n = m \); physical systems are practically always proper (but an ideal controller might be nonproper).

The coefficients of the differential equation are system parameters that completely characterize the properties of the system. They can be rescaled (if desired) by multiplying (or dividing) them all by the same factor. Rescaling to get \( C = 1 \) is always possible (if \( C \neq 0 \), the system is not proper).
4.4 Principle of superposition

For linear systems, the principle of superposition applies. Assume that the input is a nonlinear function of time, and consider one input at a time. The combined effect of several inputs is the sum of the individual effects.

The principle of superposition states that the response of a linear system to the sum of two or more inputs is equal to the sum of the responses to each input considered separately. This principle is useful for solving linear differential equations and analyzing the behavior of linear systems.


The Laplace transform is defined by the integral

\[ \mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt \]

with \( f(t) \) is a function of time that is defined for \( t \geq 0 \), \( s \) is a complex number, and \( F(s) \) is the Laplace transform of \( f(t) \). This integral is often referred to as the transform pair

\[ (f(t), F(s)) \]

where \( f(t) \) is the time-domain function and \( F(s) \) is the frequency-domain function.

From this definition, we can see that the Laplace transform is a linear transformation that changes the domain of the function from the time domain to the frequency domain. It is a powerful tool in control systems and signal processing.

In order for the Laplace transform to be useful, it is necessary that we can transform the other way, i.e. to calculate the time function from the Laplace function. This is done by using the inverse Laplace transform, which is defined as

\[ \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} e^{st} F(s) \, ds \]

where \( F(s) \) is the Laplace function and \( f(t) \) is the time-domain function.

The inverse Laplace transform is a linear transformation that changes the domain of the function from the frequency domain to the time domain. It is a powerful tool in control systems and signal processing.

The Laplace transform and its inverse are used extensively in control systems and signal processing because they allow us to simplify complex systems by transforming them into simpler algebraic expressions.

The Laplace transform is a powerful tool in control systems and signal processing because it allows us to simplify complex systems by transforming them into simpler algebraic expressions. It is used to analyze the behavior of systems in the frequency domain, which is often easier than analyzing them in the time domain.

In order for the Laplace transform to be useful, it is necessary that we can transform the other way, i.e. to calculate the time function from the Laplace function. This is done by using the inverse Laplace transform, which is defined as

\[ \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} e^{st} F(s) \, ds \]

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The inverse Laplace transform is a linear transformation that changes the domain of the function from the frequency domain to the time domain. It is a powerful tool in control systems and signal processing.
Exponential function is defined

\[
 f(t) = e^{at}
\]
where \( a \) is the arbitrary constant.

The Laplace transform can be derived from the definition (4.6)

\[
 \mathcal{L}\{f(t^m)\} = \frac{1}{s^m} F(s)
\]

We could have derived (4.15) more easily by observing that

\[
 \mathcal{L}\{e^{at}\} = \frac{1}{s-a}
\]

Since \( a > 0 \), the function is exponentially increasing.

An exponential function is defined by

\[
 f(t) = e^{-at}
\]
where \( a > 0 \). If \( t \geq 1 \), \( e^{-at} \) approaches 0.

The Laplace transform of \( e^{-at} \) is

\[
 \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}
\]

The Laplace transform of \( t e^{-at} \) is

\[
 \mathcal{L}\{t e^{-at}\} = \frac{1}{(s+a)^2}
\]

Similarly, as above, we can derive

\[
 \frac{d^{m+2}\cos(at)}{dt^{m+2}} = (-1)^m \sin(at)
\]

We could have derived (4.15) more easily by observing that

\[
 \frac{d^{m+2}\sin(at)}{dt^{m+2}} = (-1)^m \cos(at)
\]

By applying partial integration twice to the definition (4.6), we can derive

\[
 \mathcal{L}\{e^{-at} t^m\} = \frac{d^m}{ds^m} \frac{1}{s+a}
\]

The Laplace transform can be derived using integration by parts.

Mathematically:

The step function is the time derivative of the ramp function.

Consider the unit impulse, unit step, and unit ramp in Fig. 4.2.
In general, the Laplace transform of an exponentially damped time function is given by
\[ \mathcal{L}\{e^{-at}f(t)\} = \frac{(s+a)F(s)}{s} \]

where \( F(s) \) is the Laplace transform of the initial state of the system, or the initial value of the function.

For a multiple integral, consisting of \( n \) integrals, the Laplace transform is
\[ \mathcal{L}\{f(t)\} = \left(\mathcal{L}\{f(t)\}\right)^n \]

The initial value theorem states that
\[ \lim_{s \to \infty} sF(s) = f(0) \]

and the final value theorem states that
\[ \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) \]

if the initial and final steady-state values of the function whose Laplace transform is known, without doing an inverse Laplace transformation, are known. These theorems are useful for calculating initial and final steady-state values of a time function whose Laplace transform is known.
variables indicate deviations from the reference state.

where the operations only allow for a given input to be obtained by algebra. 

Thus, working in the Laplace domain, the solution of a DE, i.e.,

\[ \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \]

is the transfer function of the system described by the DE (4.27).

\[ G(s) = \frac{Y(s)}{X(s)} \]

where

\[ Y(s) = \mathcal{L}\{y(t)\}, \quad X(s) = \mathcal{L}\{x(t)\} \]

The transfer function of the system described by the DE (4.27), where all initial values of \( x(t) \) and \( y(t) \) are zero.

Thus, the Laplace transform of an operation yields a factor to the DE can be solved.

where

\[ \mathcal{L}\{y(t)\} + 2s\mathcal{L}\{y(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{0\} \]

Equation (4.29) can also be written as

\[ \mathcal{L}\{y(t) + 2\frac{dy}{dt} + y\} = \mathcal{L}\{0\} \]

Considering that the initial values of the variables are zero, the Laplace transform of an initial value yields a factor to the DE can be solved.

\[ \mathcal{L}\{y(t)\} + 2s\mathcal{L}\{y(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{0\} \]

Substitution of (4.26) into the DE (4.1), cancelling out of the reference state (which satisfies the DE separately), and the choice \( a = 0 \), yields

\[ \mathcal{L}\{y(t) + 2\frac{dy}{dt} + y\} = \mathcal{L}\{0\} \]

This point is often a steady state or an operating point, where all time derivatives are zero.

If this solution is of particular interest, we can call it a steady state or an operating point.

Assume that the DE (4.1) is satisfied by the variable values

\[ \mathcal{L}\{y(t)\} + 2s\mathcal{L}\{y(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{0\} \]

Determine the Laplace transform of the function

\[ f(t) \]

by means of the initial and final value theorems.
where the time argument

\( \tau \)

is the temperature of mercury inside the thermometer.

\( T_{\text{m}} \)

is the temperature outside the thermometer and

\( T_{\text{o}} \)

is the temperature of mercury inside the thermometer.

\( \tau_{\text{m}} \)

is the time constant of the thermometer.

\( T_{\text{m}} \)

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is the temperature of mercury inside the thermometer.

\( \tau_{\text{m}} \)

is the time constant of the thermometer.

The use of \( \zeta \) and \( \omega_n \) is when \( s = j\omega \).

The transfer function of a first-order system

\( G(s) \)

is the transfer function of a first-order system.

\( G(s) \)

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is the transfer function of a first-order system.
In the modelling of series-connected systems, it is important to know if the systems are interacting or non-interacting. A series connection is a common type of block interconnection. As for non-interacting systems, the liquid tank in the figure can be described by the transfer function $\frac{1}{s}$. However, because the liquid tank series in the figure is the liquid-tank series in the figure, the equation (4.29) results.

According to (4.29), the connection yields

\[
g = \frac{1 + s^2 T_2}{1 + s^2 T_1}
\]

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According to (4.29), the connection yields

\[
g = \frac{1 + s^2 T_2}{1 + s^2 T_1}
\]
**Process Control Laboratory**

A parallel connection is illustrated in Fig. 4.5. It contains a signal branch and a summation. Standard block algebra yields

\[ G(s) = G_1(s) + G_2(s) \]

from which it follows that

\[ G(s) = G_1(s) + G_2(s) \]

is the transfer function for a parallel connection of two subsystems.

An inverse-response system (Section 5.5) is often obtained by a parallel connection of two systems.

**Process Control Laboratory**

The most fundamental system interconnection in control theory is (negative) feedback, which is illustrated in Fig. 4.6. Standard block algebra yields

\[ G(s)H(s) = G_1(s) + G_2(s) \]

Thus, the transfer function of a feedback connection, where \( G(s) \) is the transfer function in the "forward" direction and \( R(s) \) in the feedback part.

The equation

\[ \text{characteristic equation of the system.} \]

\[ \text{The product} \]

\[ \text{The equation} \]

\[ \text{The condition is satisfied automatically when input variables are used} \]

because the input variables denote deviations from a reference state that applies at \( t = 0^- \), when the zero is approached from the negative side.

By applying the Laplace transform to a DE, such a linear expression is obtained only if the initial values of the signal and derivatives are zero. By applying the Laplace transform to a DE, such a linear expression is obtained if the initial values of the signal and derivatives are zero — no other terms can be included in the expression if there is one input.

The condition is satisfied automatically when input variables are used because they denote deviations from a reference state that applies at \( t = 0^- \), when the zero is approached from the negative side.

Since it is an essential requirement in calculations with transfer functions that the signals have the above characteristics, it is considered to be the case even if it is not mentioned. Thus, as shown in Exercise 4.4, the symbol \( \dot{\cdot} \) can be omitted to simplify the notation.

If \( \dot{\cdot} \)-variables with the symbol \( \dot{\cdot} \) are used, it is often to emphasize the physical meaning of the symbol. For example, \( \dot{x} \) is used to denote the first derivative of a variable, \( \ddot{x} \) the second derivative, and so on. The symbol \( \dot{\cdot} \) is used to indicate a rate or a change, and it is often used in engineering to denote the time derivative of a quantity.

A measurement in the process.
It is recommended to denote time functions with "small" letters (lower-case letters) and their Laplace transforms with the corresponding "large" letter (capital letter). However, it is not unusual to use the same symbol for both the time function and its Laplace transform; sometimes this is due to lack of available letters. This can be done because it is usually clear from the context which the function type is. For example, in calculations with transfer functions, it is clear that the Laplace transform of the signals are used.

If there is a danger of misunderstanding, the argument \( t \) or \( s \) can be included to indicate the type of function. This may be needed if the same symbol is used for the time function and its Laplace transform.

It should be noted that time-domain signals and their Laplace transform generally have a physical unit. Mathematical operations should therefore satisfy unit dimensions both in the time domain and the Laplace domain.

In particular, it should be noted that the gain of a system is not dimensionless if the input and the output have different units.

The inverse Laplace transform is used when we want to find the time-domain function that corresponds to a given Laplace-domain function. The general procedure for solving time-domain problems in the Laplace domain is illustrated in the figure.

A convenient way to solve linear ordinary differential equations is to use Laplace transform methods. When the Laplace transform is applied to a differential equation term by term, the Laplace transform of the dependent variable, i.e., the output signal, can be solved by purely algebraic methods. If the differential equation describes a dynamic system, it has an input that is also transformed. The time function of the dependent variable can then be obtained by using the inverse transform of its Laplace transform.

The superposition theorem (see Section 4.2.3) then allows us to obtain the full time function as the sum of the time functions of the simpler Laplace functions.

### Example

Solve the differential equation

\[ \frac{d^2}{dt^2} x(t) + 5 \frac{dx}{dt} + x(t) = 2 \]

with the initial values \( x(0^-) = 0 \) and \( \frac{dx}{dt}(0^-) = 1 \).

The Laplace transform of the DE with the given initial values is obtained by (4.18) and (4.19) as

\[
\mathcal{L}\{x\} - x(0^-) + 5\mathcal{L}\{x\}'(0^-) + \mathcal{L}\{x\} = \frac{2}{s}
\]

Substitution of the initial values yields

\[
\mathcal{L}\{x\} - 0 + 5\mathcal{L}\{x\}'(0^-) + \mathcal{L}\{x\} = \frac{2}{s}
\]

or

\[
\mathcal{L}\{x\} = \frac{2}{s} - 5\mathcal{L}\{x\}'(0^-)
\]

According to the superposition theorem (see Section 4.2.3), the full time function is then obtained as

\[
x(t) = \frac{2}{s} - 5\mathcal{L}\{x\}'(0^-)
\]

The solution should be verified by checking if it satisfies the DE + i.c.'s.

### Diagram

- **Problem Statement:**
  - The problem involves solving a time-domain problem using the Laplace transform.
  - The initial conditions are given, and the Laplace transform of the differential equation is solved.

- **Solution Process:**
  - The Laplace transform is applied to the differential equation, resulting in an algebraic equation.
  - The inverse Laplace transform is used to obtain the time-domain solution.
  - The solution is verified to ensure it satisfies the original differential equation.

- **Note:**
  - The superposition theorem is used to find the full time function as the sum of the simpler Laplace functions.

- **Key Concepts:**
  - Time-domain signals and their Laplace transforms.
  - Dimensionality considerations.
  - Physical units in mathematical operations.
  - Use of the inverse Laplace transform to find time-domain solutions.
The inverse transform of \( g(t) \) can be obtained via a PFE. Let us substitute an impulse and the derivatives of impulses of \( f(s) \) will consist of one or more

\[
\begin{align*}
\mathcal{L}^{-1}\{f(s)\} &= g(t) \\
\mathcal{L}^{-1}\{g(t)\} &= f(s)
\end{align*}
\]

According to the superposition principle, the inverse Laplace transform of

\[
\mathcal{L}^{-1}\{f(s)\} + \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\{f(s) + g(s)\}
\]

is the sum of the inverse transforms of \( f(s) \) and \( g(s) \). The inverse transform \( \mathcal{L}^{-1}\{f(s)\} \) of the inverse Laplace transform of

\[
\mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt
\]

is the solution of the differential equation

\[
f(t) = \mathcal{L}^{-1}\{F(s)\}
\]

where \( F(s) \) is the Laplace transform of \( f(t) \). This solution is a polynomial of order \( n \) or \( n-1 \) if \( f(t) \) is the derivative of \( f(t) \). Note the similarities and differences with Example 4.4.

According to pt 1 in the Laplace table (and the superposition principle), the inverse Laplace transform of \( \mathcal{L}\{f(t)\} \) is the sum of the inverse transforms of \( f(t) \) and \( g(t) \) of \( \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \).

For a Laplace function \( \mathcal{L}\{f(t)\} \), the time function \( f(t) \) can be determined from \( \mathcal{L}\{f(t)\} \) by performing a partial fraction expansion

\[
\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1}
\]

and \( f(t) \) is obtained by performing the inverse Laplace transform of \( \mathcal{L}\{f(t)\} \). This is done by performing a partial fraction expansion of \( \mathcal{L}\{f(t)\} \) to obtain

\[
\begin{align*}
\mathcal{L}\{f(t)\} &= \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1} \\
&= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{s^2 + 2s + 1}
\end{align*}
\]

where \( A, B, C, \) and \( D \) are constants.

For a Laplace function \( \mathcal{L}\{f(t)\} \), the time function \( f(t) \) can be determined from \( \mathcal{L}\{f(t)\} \) by performing a partial fraction expansion

\[
\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1}
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&= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{s^2 + 2s + 1}
\end{align*}
\]

where \( A, B, C, \) and \( D \) are constants.

4.4.1 Solving Linear DEs

Example 4.3: Step Response of a First-Order System

\[
\begin{align*}
\mathcal{L}\{f(t)\} &= \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1} \\
&= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{s^2 + 2s + 1}
\end{align*}
\]

where \( A, B, C, \) and \( D \) are constants.

For a Laplace function \( \mathcal{L}\{f(t)\} \), the time function \( f(t) \) can be determined from \( \mathcal{L}\{f(t)\} \) by performing a partial fraction expansion

\[
\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1}
\]

and \( f(t) \) is obtained by performing the inverse Laplace transform of \( \mathcal{L}\{f(t)\} \). This is done by performing a partial fraction expansion of \( \mathcal{L}\{f(t)\} \) to obtain

\[
\begin{align*}
\mathcal{L}\{f(t)\} &= \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s-1} + \frac{4}{s^2 + 2s + 1} \\
&= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} + \frac{D}{s^2 + 2s + 1}
\end{align*}
\]

where \( A, B, C, \) and \( D \) are constants.
are not very common.

are the constants that need to be determined.

\[ A_1, A_3, A_4 = 0, \quad d_1 = d_3 = 0 \]

\[ d_2 = d_4 = 1 \]

\[ f(\sigma) \] in Eq. (4.4) can be cancelled out against the same factor in the denominator

If the roots are distinct and real, \( A_1, A_3, \) can be determined according to

The form of the PFE depends on the properties of the roots

where \( A_1, A_3, A_4 = 0, \quad d_1 = d_3 = 0 \)

\[ d_2 = d_4 = 1 \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

\[ f(\sigma) \]

\[ \sigma \]

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The ramp response of a system is the output as a function of time when the input changes as a ramp, i.e., linearly with time. The input is thus

\[ u(t) = \alpha t \]

where \( \alpha \) is the slope of the input ramp. The function has the Laplace transform (pt 2 in the Laplace transform table)

\[ \mathcal{L}\{u(t)\} = \frac{\alpha}{s} \]

The Laplace-domain output of a first-order system is given by

\[ \mathcal{L}\{y(t)\} = \frac{1}{\tau s + 1} \]

Substitution of (2) into (3) yields

\[ \mathcal{L}\{y(t)\} = \frac{1}{\tau} \mathcal{L}\{u(t)\} \]

The time-domain output

\[ y(t) = \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) \]

The expression must apply separately for each power of \( s \):

\[ \begin{align*}
    s^0 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right) \\
    s^1 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right) \\
    s^2 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right)
\end{align*} \]

Since \( e^{-\frac{t}{\tau}} \to 0 \) as \( t \to \infty \), the output approaches a ramp with the ramp coefficient \( \tau \alpha \).

The time-domain output \( y(t) \) is given by the inverse Laplace transform.

The expression applies separately for each power of \( s \):

\[ \begin{align*}
    s^0 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right) \\
    s^1 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right) \\
    s^2 & : \quad \tau \frac{\alpha}{\tau} \left( 1 - e^{-\frac{t}{\tau}} \right) = \alpha \left( 1 - e^{-\frac{t}{\tau}} \right)
\end{align*} \]
5.1 Introduction to Simple Systems

5.2 First-Order Systems

5.3 Second-Order Systems

5.4 Time-Delay Systems

5.5 Inverse-Response Systems

S.1 Transient Response

5.6 Time-Delay Systems

S.2 Introduction to Simple Systems

5.7 Table of Laplace Transforms

5.8 Table of Laplace Transforms

\[ \frac{1}{s^2 + 2\omega_0 s + \omega_0^2} \]

\[ \frac{1}{s^2 + \omega_0^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]

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\[ \frac{1}{s^2 + \omega_n^2} \]

\[ \frac{1}{s^2 + \omega_n^2} \]
In Chapter 3, we derived models for some simple technical systems. In all cases, the systems could be described by ordinary differential equations (ODEs) of relatively low order. In many cases, the ODEs were nonlinear, but they can be linearized around an operating point. In this chapter, we shall study the properties of some simple, linear, dynamical systems.

In particular, we shall derive the time response to well-defined inputs such as impulses and steps; analysis based on such responses is called transient analysis. We shall also study simple, graphical techniques for determining linear low-order models from step-response data; these are simple examples of system identification.

An integrating system is the simplest dynamical system that can be described by a differential equation. The most typical process example of such a system is probably a liquid tank.

Consider the liquid tank in Fig. 5.1.

\[ \begin{align*}
\frac{dL}{dt} &= \frac{s}{A} = \frac{\frac{dc}{dt}}{A} \\
\frac{dc}{dt} &= \frac{Q_1}{A} - \frac{Q_2}{A}
\end{align*} \]

The Laplace transform, considering that the initial state is zero, yields

\[ \begin{align*}
\frac{c}{s} &= \frac{Q_1}{s} - \frac{Q_2}{s} \\
&= \frac{Q_1}{s} - \frac{Q_2}{s} \\
&= \frac{Q_1}{s} - \frac{Q_2}{s}
\end{align*} \]

The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]

which, according to the Laplace transform, corresponds to integrals in the time domain. The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]

The two transfer functions of the system are

\[ G_1(s) = \frac{Q_1}{s} \quad \text{and} \quad G_2(s) = \frac{Q_2}{s} \]

We shall also study the response to an input signal and output signal. Generally, a linear integrating system with input signal and output signal can be described by the differential equation

\[ \frac{dL}{dt} = \frac{s}{A} = \frac{dc}{dt} \]

The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]

which, according to the Laplace transform, corresponds to integrals in the time domain. The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]

The two transfer functions of the system are

\[ G_1(s) = \frac{Q_1}{s} \quad \text{and} \quad G_2(s) = \frac{Q_2}{s} \]

We shall also study the response to an input signal and output signal. Generally, a linear integrating system with input signal and output signal can be described by the differential equation

\[ \frac{dL}{dt} = \frac{s}{A} = \frac{dc}{dt} \]

The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]

which, according to the Laplace transform, corresponds to integrals in the time domain. The transfer function of the system is

\[ G(s) = \frac{Q_1}{s} - \frac{Q_2}{s} \]
A non-integrating first-order system can be described by the differential equation

\[ \frac{dy}{dt} + \lambda y = \frac{N}{\Delta} \Delta t \]

where \( \lambda \) is the time constant and \( N \) is the static system gain. The transfer function of the system is

\[ G(s) = \frac{N}{\lambda s + 1} \]

The time response of the system output to a given input signal is called a response. It can be determined by finding the inverse Laplace transform of the transfer function. Two common inputs for transient analysis are the (unit) impulse function and the step function.

An input impulse with the time integral (area, volume, mass, etc.), i.e.,

\[ y(t) = \delta(t) \]

where \( \delta(t) \) is the unit impulse (Dirac delta function), has the Laplace transform (LT Table, pt 6)

\[ Y(s) = \frac{N}{s} \]

Substitution into (5.4) gives

\[ \frac{1}{s} Y(s) = G(s) \]

The inverse Laplace transform (LTT, pt 25) gives the impulse response

\[ y(t) = \frac{N}{\lambda} e^{-\lambda t} \]

An input step change of size \( N \Delta t \), i.e.,

\[ y(t) = N \Delta t \delta(t) \]

where \( \delta(t) \) is the unit step, has the Laplace transform (LT Table, pt 1)

\[ Y(s) = \frac{N \Delta t}{s} \]

Substitution into (5.4) gives

\[ \frac{1}{s} Y(s) = G(s) \]

The inverse Laplace transform (LTT, pt 26) gives the step response

\[ y(t) = \frac{N \Delta t}{\lambda} e^{-\lambda t} \]

The initial slope (i.e., the derivative) of the responses is the slope of a line drawn from the start of the response to the line denoting the new steady state at time \( t = \lambda \).

The vertical distance between the response and the point of intersection at \( t = \lambda \) is \( \frac{N \Delta t}{\lambda} \) of the total change of the output. In practice, the new steady-state value (within 2%) is reached at \( t = \lambda + \lambda \) (but in theory, it takes a longer time).
The step response of a first-order system reaches 63.2% of the total change at the time $t = \tau$ (see Fig. 5.3). Thus, the time constant $\tau$ can be estimated as the time it takes for the step response of a first-order system to reach 63.2% of the total change. This estimate is often used even when the true system is not a first-order system. It is then called the equivalent time constant $\tau$. In practice, a system often contains a time delay. A first-order system with a time delay $\tau_d$ has the transfer function $G(s) = \frac{1}{\tau_d s + 1}$ (5.10) and step response $y(t) = \left(1 - e^{-t/\tau} / (1 - e^{-\tau_d/\tau})\right)$ (5.11). The time delay $\tau_d$ is estimated as the time until the outputs starts to respond (input change at $t = 0$). The time to reach 63.2% of the total change gives $\tau + \tau_d$.

In practice, a system is hardly ever a perfect first-order system. Often the step response does not have its steepest slope immediately at the beginning of the response. This means that the system is of higher order than first order. In spite of this, we might be interested in approximating the system as a first-order system with a time delay described by (5.10). The static gain is calculated by Eq. (5.9). To determine the time delay and time constant, a tangent through the inflection point (i.e., the point of the steepest slope) of the step response is drawn. The point of intersection with the time axis gives the time delay $\tau_d$, the final asymptotic value gives the time constant $\tau$. Thus, the true step response --- step response of model --- The tangent method

Comments on the tangent method
For model identification
As shown by Fig. 5.5, the step response of the identified model can differ considerably from the true step response; the model response will always be slower than the true response. This means that the time constant determined by the tangent method will always be too large. Thus, the tangent method is not a good method for estimating an approximate 1st-order time constant. For controller tuning
There are controller tuning methods, e.g., Ziegler-Nichols's step-response base recommendations, that assume that the tangent method has been used to characterize the dynamics. In such cases, it is OK to use the tangent method, because the time delay is determined by the tangent method. The time constant is determined according to the 63% method such that $\tau + \tau_d$ is the time it takes to reach 63.2% of the total output change.

This procedure is better than the pure tangent method, because it gives a model whose step response matches the true step response much better; it is more robust against disturbances and measurement errors because two response points are used instead of only one.
5.3 Transient Response

The inflection point and the point where the step response reaches 63.2% of the total change are often close to each other. It would be better to have a method where the two points are farther apart. In a method developed by Sundresan and Krishnaswamy, the points at 35% and 85% of the total output change are used.

According to the K-S method, the time constant and time delay are calculated by

\[
\begin{align*}
\tau &= \frac{1}{\omega_n} \\
p &= \frac{1}{\tau} \\
\end{align*}
\]

where \(\omega_n\) and \(p\) are the times it takes to reach 35% and 85% of the total output change. According to the K-S method, the time constant and time delay are calculated by

\[
\begin{align*}
\tau &= \frac{1}{\omega_n} \\
p &= \frac{1}{\tau} \\
\end{align*}
\]

where \(\omega_n\) and \(p\) are the times it takes to reach 35% and 85% of the total output change.
The transfer function of an overdamped system, without a time delay, is usually written in the form

\[ G(s) = \frac{\frac{1}{\tau_1 \tau_2} (\tau_1 + \tau_2)}{s + \frac{1}{\tau_1} + \frac{1}{\tau_2}} \]

where

\[ \tau_1 = \tau_1 \] and \[ \tau_2 = \tau_2 \] according to

\[ \tau_1 / \tau_2 = 1 \] and \[ \tau_2 / \tau_1 = 1 \] (5.17)

The system has the impulse response

\[ y(t) = \frac{1}{\tau_1} \] (input \( u(t) \))

and the step response

\[ y(t) = \frac{1}{\tau_1} \] (5.18)

The responses are shown in Fig. 5.8 and 5.9.

The transfer function of a critically damped system, without a time delay, is often written in the form

\[ G(s) = \frac{1}{s^2 + \omega_n^2} \]

\[ \omega_n = \omega_n \]

\[ \tau_1 = \tau_1 \]

\[ \tau_2 = \tau_2 \] (5.22)

The system has the impulse response

\[ y(t) = \frac{1}{\tau_1} \] (5.23)

and the step response

\[ y(t) = \frac{1}{\tau_1} \] (5.24)

The responses are shown in Fig. 5.8 and 5.9.

The transfer function of an underdamped system, without a time delay, is usually represented in the form (5.15).

Because the roots of the characteristic equation are complex, the analytical expressions for transient responses contain trigonometric functions.

The impulse response

\[ y(t) = \frac{1}{\tau_1} \] (input \( u(t) \))

and the step response

\[ y(t) = \frac{1}{\tau_1} \] (5.25)

where

\[ \beta = \beta \]

\[ \tau = \tau \] (5.26)

The step response

\[ y(t) = \frac{1}{\tau_1} \] (input \( u(t) \))

and the step response

\[ y(t) = \frac{1}{\tau_1} \] (5.27)

where

\[ \gamma = \gamma \]

(5.28)

The responses are shown in Fig. 5.8 and 5.9.

Figures 5.8 shows the impulse response and Fig. 5.9 the step response for second-order systems without zeros. A time delay would not affect the form of the responses, only delay them. When the output and the time are normalized as shown in the figures, the form of the responses, only delay them.

The figures show that overdamped and critically damped systems have monotonic responses, whereas underdamped systems exhibit oscillatory responses. The transfer functions are parameterized by the relative damping ratio, and the responses are shown in the figures. The responses are shown in Fig. 5.8 and 5.9.
Control of Simple Systems
5.3 Second-order systems

5.3.2 Identification of overdamped systems

5.3.2 Identification of overdamped system

A simple method for the identification of an overdamped second-order system

The static gain is the ratio of the output change at a time to the input change. It is unknown initially, so it must be determined by the identification procedure. Alternatively, it can be calculated from the step response (5.8).

### Identification procedure

When the system is to be identified, it is assumed that the time constant and the static gain are known. The initial value of the output is zero.

\[ y(0) = 0 \]

The time delay is determined by the tangent method.

### Identification of a second-order system

The static gain is calculated by (5.30).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the time difference between the initial change and the steady-state response (5.31).

### Identification of overdamped systems

The static gain is calculated by (5.32).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.33).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.34).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.35).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.36).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.37).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.38).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.39).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.40).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.41).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.42).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.43).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.44).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.45).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.46).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.47).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.48).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.49).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.50).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.51).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.52).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.53).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.54).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.55).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.56).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.57).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.58).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.59).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.60).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.61).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.62).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.63).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.64).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.65).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.66).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.67).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.68).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.69).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.70).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.71).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.72).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.73).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.74).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.75).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.76).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.77).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of first-order systems

The static gain is calculated by (5.78).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of second-order systems

The static gain is calculated by (5.79).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.

### Identification of overdamped systems

The static gain is calculated by (5.80).

\[ K = \frac{y(t_f)}{u(t_f)} \]

The time delay is determined by the tangent method.
Determine $\tau_2$ from Fig. 5.12, or calculate it by

$$\tau_2 = \frac{b}{A}$$

(5.33)

Calculate the sum of the time constants by

$$\tau = \tau_1 + \tau_2$$

(5.34)

Calculate the time constants by

$$\tau_1 = \frac{1}{\frac{1}{\tau_1} + \frac{1}{\tau_2}}$$

(5.35a)

$$\tau_2 = \frac{1}{\frac{1}{\tau_1} + \frac{1}{\tau_2}}$$

(5.35b)

Now all parameters of the transfer function (5.29) have been determined!
Identification of 1st order system

The time delay is determined by the intersection of the tangent through the inflection point of the step response with the time axis.

The time constant is determined as the time to reach 63.2% of the total output change reduced by the time delay.

The transfer function is given by:

\[ G(s) = \frac{7}{s+7} \]

Fig. 3 shows this step response together with the original step response. As can be seen, the fit is not so good.

If the model parameters are determined by numerical optimization over the whole visible step response, the result is:

\[ T = 7 \text{ and } K = 1.3 \]

The step response is shown in Fig. 6. The overall fit is better, but clearly the time delay seems too large (and \( K < 5 \)).

For the input step \( \xi(t) \), the identified system has the step response:

\[ y(t) = \frac{7}{s+7} (1 + e^{-s/7}) \]

Fig. 6 shows this step response together with the step response.

Identification of 2nd order system

The time delay should be shorter than in the identification of a 1st order system since the "smooth" start of the step response can be handled by the 2nd order dynamics. However, it must be long enough to satisfy (5.31).

This suggests the choice:

\[ T = 7 \text{ and } K = 1.3 \]

This generates the transfer function:

\[ G(s) = \frac{1+7s}{s^2+7s+9} \]

The step response is shown in Fig. 5. As can be seen, the fit is almost perfect.

If the model parameters are determined by numerical optimization over the whole visible step response, the result is:

\[ T = 7 \text{ and } K = 1.3 \]

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The step response is shown in Fig. 5. As can be seen, the fit is almost perfect.
Step Response Parameters

5.3 Step Response Parameters of Underdamped System

- Maximum relative overshoot: \( M_r = \frac{\max y(t) - y(\infty)}{y(\infty)} \)
- Period of oscillation (first period is easiest to measure): \( T = \frac{2\pi}{\omega_n} \)
- Maximum relative overshoot: \( M_p = \max y(t) \)
- Time to first peak: \( t_p = \frac{1}{\omega_n} \ln \left( \frac{1}{1-M_r} \right) \)
- Rise time: \( t_r = \frac{1}{\omega_n} \ln \left( \frac{1}{\frac{1}{2} - M_r} \right) \)
- Settling time: \( t_s = \frac{1}{\omega_n} \ln \left( \frac{1}{0.01} \right) \)

Note: The expressions for these parameters are exact relationships, whereas (5.39) is only approximate. The expressions for (5.38) are exact relationships when \( \omega_n \) is the undamped natural frequency.

The rise time \( t_r \) is the time required for the output to reach and remain between the initial value of the output signal and the final value of the step response.

The settling time \( t_s \) is the time required for the output to reach and remain within 1% of the final value.
The identification of an underdamped 2nd-order system is illustrated by use of the step response in Fig. 5.13. Because of missing data, the following assumptions are used:

- The time axis goes from 0 to 20 seconds.
- The output signal axis goes from 0 to 2.5.

The figure yields:

\[ y(t) = e^{-\frac{t}{\tau}} - e^{-\frac{t}{\tau_1}} \]

(1)

Equation (5.26) and (5.35) can be solved with respect to \( \zeta \) to yield:

\[ \zeta = \sqrt{1 - \frac{\omega_n^2}{\omega_d^2}} \]

(3)

Numerically, (3) yields \( \zeta = 0.7 \). Eq. (5.36) yields \( \omega_d = 1 \). The gain \( \zeta \) cannot be determined since the size of the input step is not given.

The true values used to calculate the step response are \( \zeta = 0.06 \) and \( \omega_d = 0.93 \).

In practice, a time delay is often caused by a transport delay. A conveyor belt is a typical example of a transport delay. For liquid and gas flows in a pipeline, the properties of the flowing medium (e.g., temperature and concentration) are delayed when the output is located some distance from the input. However, the mass flow rate of an incompressible fluid is not delayed.

Measurement instruments can sometimes cause a time delay. For example, a time delay occurs caused by a transport delay.

The exponential function is an irrational function. This causes analysis problems when it is combined with a rational function (polynomial). For this reason, it is often desirable to approximate the exponential function by a rational expression.

Simple rational approximations of the exponential function can be derived via Taylor series expansions. A straightforward expansion of \( e^{-\frac{t}{\tau}} \) gives:

\[ e^{-\frac{t}{\tau}} \approx 1 - \frac{t}{\tau} \]

(5.43)

This expression is exact when infinitely many terms are included, but useful approximations are obtained by a finite number of terms.

The first two terms of (5.43) give the simple, but relatively inexact, first-order Taylor series approximation:

\[ e^{-\frac{t}{\tau}} \approx 1 - \frac{t}{\tau} \]

(5.44)

Using more terms will improve the approximation, but this increases the order of the resulting polynomial when the approximation is combined with another polynomial expression. Another class of approximations is obtained by a Taylor series expansion of the exponential function.

Another example of a non-minimum phase system is an inverse-response system (Section 5.5). This is a system with positive zeros. The exponential function is an irrational function, and it is often desirable to approximate it by a rational expression.

Simple rational approximations of the exponential function can be derived via Taylor series expansions. A straightforward expansion of \( e^\frac{t}{\tau} \) gives:

\[ e^\frac{t}{\tau} \approx 1 + \frac{t}{\tau} \]

(5.45)

A first-order approximation of \( e^\frac{t}{\tau} \) yields:

\[ e^\frac{t}{\tau} \approx 1 + \frac{t}{\tau} \]

(5.46)

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A first-order approximation of \( e^\frac{t}{\tau} \) yields:

\[ e^\frac{t}{\tau} \approx 1 + \frac{t}{\tau} \]

(5.46)
The Taylor series expansions (5.43) and (5.45) can be combined to yield

\[
\begin{align*}
\sum_{k=0}^{\infty} & \frac{a_k}{s^k} = 0 \\
\sum_{k=0}^{\infty} & \frac{b_k}{s^k} = 0
\end{align*}
\]

where \( a_k \) is any parameter in the range \( 0 < \frac{a}{b} < 1 \) and \( b_k \) is obtained by solving that \( g \) is a non-minimum time constant in each so

\[
\begin{align*}
t & = \sum_{k=0}^{\infty} \frac{a_k}{s^k} \\
t & = \sum_{k=0}^{\infty} \frac{b_k}{s^k}
\end{align*}
\]

Using \( a = 0 \) and a first-order approximation of the numerator and the denominator gives

\[
\begin{align*}
t & = \sum_{k=0}^{\infty} \frac{a_k}{s^k} \\
t & = \sum_{k=0}^{\infty} \frac{b_k}{s^k}
\end{align*}
\]

where \( a_k \) is any parameter in the range \( 0 < \frac{a_k}{b_k} < 1 \) and \( b_k \) is obtained by solving that \( g \) is a non-minimum time constant in each so

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\end{align*}
\]
6. Stability

6.1 Stability definitions

6.1.1 Asymptotic stability

A system is asymptotically stable if it returns to its initial state after a transient disturbance. A typical transient disturbance is a pulse, and in practice, many calculations become easier if the pulse is assumed to be an impulse.

Remark 1. Asymptotic stability is often defined in more mathematical terms than the above. Although the definitions might seem different, they are equivalent.

Remark 2. It follows from the definition that an input-output stable system has a finite gain at all frequencies (see Chapter 8).

6.1.2 Input-output stability

A system is input-output stable if limited input signal results in a limited output signal. A typical limited input signal is a step change.

Remark 2. A step change is not a transient disturbance.

6.2.1 Input-output stability

Input-output stability in terms of system poles can be found by direct substitution.

6.3 Analysis methods

6.3.2 Ljapunov's stability criterion

Stability can be defined in many different ways. For all practical purposes, the different definitions are equivalent for linear systems. In a given situation, a certain definition may be more convenient to use than other definitions. The two following quite concrete definitions are general in so far as they are independent of the type of system description (transfer function or state-space model).

6.2 Stable and unstable systems

6.2.1 The time response of a linear system

As shown in the first two chapters, a successful controller design requires a compromise between performance ("speed") and stability. We can conclude that stability is necessary, but not sufficient, for good control. It is obvious that systematic methods are needed to determine if a system — controlled or uncontrolled — is stable or unstable.

6.1.1 Asymptotic stability

An uncontrolled system that is stable may become unstable by aggressive control. On the other hand, there are also systems that are unstable without control which can be stabilized by control. Stability can be defined in many different ways. For all practical purposes, the different definitions are equivalent for linear systems. In a given situation, a certain definition may be more convenient to use than other definitions. The two following quite concrete definitions are general in so far as they are independent of the type of system description (transfer function or state-space model).

6.3 Analysis methods

6.3.2 Lyapunov's stability criterion

Stability can be defined in many different ways. For all practical purposes, the different definitions are equivalent for linear systems. In a given situation, a certain definition may be more convenient to use than other definitions. The two following quite concrete definitions are general in so far as they are independent of the type of system description (transfer function or state-space model).
The input signal to the system is the output signal of the system.

The condition for input-output stability using a step function is given by

\[ y(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \]

If the system is strictly proper, the impulse response grows at a slower rate than the transient response. If the system is strictly improper, the impulse response grows at a faster rate than the transient response.

When the characteristic polynomial \( A(s) \) has distinct or repeated roots, the system is said to be asymptotically stable. If the roots are complex-conjugated, the system is said to be stable. If any root has a positive real part, the system is said to be unstable.

The Laplace transform of the impulse response is given by

\[ \mathcal{L}\{y(t)\} = \frac{1}{s} \]

The inverse Laplace transform gives the time response of the system.

\[ y(t) = \frac{1}{s} \Rightarrow y(t) = u(t) \]

where \( u(t) \) is the unit step function.

The system is stable if and only if the poles of the characteristic polynomial are in the left half-plane (i.e., \( \text{Re}(s) < 0 \)).
If the characteristic polynomial contains repeated (multiple) zeros, either real or complex-conjugated, the following applies.

The inverse Laplace transform of the PFE will, besides similar terms as in (6.6), (6.7) and (6.8), contain products of exponential functions and the time $t$ raised to a certain power.

Because the exponential function $e^{\lambda t}$, with $\lambda = \sigma - j \omega$, decreases faster than $t^n$ grows for any power $n$, such terms as in (6.6), (6.7) and (6.8) will decay with time.

From this it follows that the stability conditions derived for distinct poles also apply when the system has repeated poles.

"..."
Show that the system is unstable. Determine if it can be stabilized by a P controller.

Is the system stable or unstable? Determine if the closed-loop system is stable when the system is controlled by a PI controller with the parameters:

(a) $K_p = 1$, $K_i = 0$
(b) $K_p = 1$, $K_i = 0.5$
(c) $K_p = 1.5$, $K_i = 0.5$

Use of the stability criterion, defined by the system poles, requires that the poles can be determined.

For systems of higher order than 2, it can be difficult or even impossible to determine the poles analytically, but if there are no unknown parameters, the poles can be calculated numerically.

Often it is of interest to investigate the stability limit as a function of one or more unknown parameters (e.g., controller parameters), preferably so that the limit can be expressed by one or more analytical expressions. This is problematic for high-order systems.

Another complication arises if the system contains a time delay which appears in the characteristic equation. This situation occurs if a feedback-controlled system has a time delay.

For these reasons, various methods for stability analysis have been developed. The following methods are studied in this course.

- The use of the Routh-Hurwitz stability criterion requires that the characteristic equation can be written as a polynomial. As noted, a possible time delay can be approximated by a rational expression, e.g., a Padé approximation. In this case, the stability analysis can be written as a polynomial.

- This method, which is studied in Section 6.3.2, can give stability intervals with respect to unknown parameter values, for example, controller parameters. High system orders cause no special problems, but time delays can be handled accurately.

- This method, which is studied in Section 6.3.3, is based on the fact that the system poles, i.e., the zeros of the characteristic equation, must be located on the imaginary axis of the complex plane at the stability limit. Stability intervals with respect to unknown parameter values can be obtained. Time delays can be handled accurately, but for systems of higher order than 2, it can be difficult or even impossible to determine the poles.
with positive real part for the time delay. If any element in the first column is non-positive, the system is unstable. The stability condition is that all the elements in the first column of the RH table are positive.

The stability criterion is that the elements in the first column of the RH table are positive.

Remark 1. The elements of row 5 are calculated according to (6.16c) so that the number of elements is one less than in two rows above.

Remark 2. If an element in the first column is equal to zero, it corresponds to a pole with the real part equal to zero. If any other element is equal to zero, the calculations can stop.

Remark 3. During the calculations, the elements become clear that all remaining elements must be equal to zero. If the calculations can stop.

Exercise 6.2

Determine if the feedback system in the figure is stable using Routh-Hurwitz's stability criterion. The system in the figure is obtained as the difference between the current and the previous rows. If an element needed in a calculation below is missing, the value zero is used.

Exercise 6.3

Show that the following stability conditions apply when the characteristic equation is in the form of (6.15) with 

\[ \det(A - sI) = 0 \]

Example is controller parameters if the system is a feedback system.

A typical example is controller parameters if the system is a feedback system.

Remark 3. The stability criterion that all elements in the first column have to be positive is used in the calculations. Since all the systems in the table are reduced systems with the same structure as in Exercise 6.6, it follows that the controller gain is reduced.

The stability condition is that the number of sign changes in the first column are equal to the number of the poles. Use a first-order Padé approximation 

\[ \frac{z}{1 + z^2} \]

when the characteristic equation is in the form of (6.15) with 

\[ \det(A - sI) = 0 \]

Laboratory Process Control
Because the poles of a stable system are located in the left-half complex plane (LHP), the imaginary axis represents the stability limit. If \( z = j \omega \) is at the stability limit, at least one pole of the system is located on the imaginary axis. Such a pole, which has the form \( z = j \omega \), is given by \( z = \lambda + j \omega \), \( \lambda \) and \( \omega \) are real numbers, and \( \omega > 0 \). These parameters at the stability limit are also obtained.

Time delays can be treated exactly, because Euler's formula

\[
\exp(j \omega T) = \cos(\omega T) - j \sin(\omega T)
\]

can be used.

Solve Exercise 6.7 by direct substitution without approximation of the time delay.

Solve Exercise 6.6 by the method of direct substitution

\[
0 = (p) 0 + (d) 0
\]

Substitution of \( s \) for \( \omega \) and \( \tau \) for \( \tau \) gives the characteristic equation

\[
0 = (p) 0 + (d) 0
\]

which can be solved to yield \( \omega = \alpha \). If \( c \) and/or \( d \) contain an unknown parameter, the value of this parameter at the stability limit is also obtained.

\[
0 = (p) 0 + (d) 0
\]

The stability limit

6.33 Finding the stability limit by direct substitution

A PID controller ("pee-i-dee") is a generic name for a controller containing a linear combination of proportional (P), integral (I), and derivative (D) terms acting on a control error (or sometimes the process output). All parts need not be present. Frequently I and/or D action is missing. It has been estimated that of all controllers in the world 95% are PID controllers. Of course, there are many variations of this type of controller. An example is the proportional-integral-derivative (PID) controller, which is composed of a proportional, an integral, and a derivative action.

It has been estimated that of all instruments of the world 95% are PID controllers. Of course, there are many variations of this type of instrument. An example is the proportional-integral-derivative (PID) instrument, which is composed of a proportional, an integral, and a derivative action.
The transfer function of a parallel form PID controller with a derivative is
\[
\frac{s^2 + sL + 1}{s^2 + \frac{1}{T} + 1} Y = G_p d
\]
for \( T \) and \( L \) are filter constants, usually 10-30\% of corresponded derivative time.

If the transfer function of a series form PID controller with a derivative is
\[
\left(1 + \frac{sL}{T} + 1\right) \left(\frac{1}{T} + 1\right) Y = G_p d
\]
where \( T \) and \( L \) are filter constants.

The transfer function of a parallel form PID controller is
\[
\frac{s^2 + sL + 1}{s^2 + \frac{1}{T} + 1} Y = G_p d
\]
and its parallel form is sometimes referred to as
\[
(7.4)
\]
A parallel form controller is obtained from a PID controller by letting \( T = \infty \).

A derivative is obtained from a PID controller by letting \( T = 0 \). It is
\[
(7.5)
\]
Complex poles are used for control of underdamped systems with
\[
(7.6)
\]
depending on the values of \( T \) and \( L \) the transfer function of the PID

The integral form of a PID controller can only have real valued zeros.

A derivative is obtained from a PID controller by letting \( T = 0 \). It is
\[
(7.7)
\]
Complex poles are used for control of underdamped systems with
\[
(7.8)
\]
depending on the values of \( T \) and \( L \) the transfer function of the PID.

Controller

Which is the control law in the time domain for a parallel form PID

\[
(7.9)
\]
\( u(t) \) is the voltage or current output of the controller.

\( e(t) \) is the load or process value that the controller must control.

\( \tau \) is the integral time.

\( \omega \) is the derivative time.

\( L \) is the integral time.

\( T \) is the derivative time.
Process Dynamics and Control

2.7.4 Separately for Setpoint Tracking and Disturbance Rejection.

A 2DOF controller can be tuned separately for setpoint tracking and disturbance rejection. This kind of controller can be tuned for setpoint tracking and disturbance rejection in the Laplace domain. The control law with setpoint weighting is in the form PID controller which is equivalent to the parallel form according to

\[ \text{Controller parameters} \]

For calculation of the parameters of the series form from the parameters of the parallel form, we define the parameter

\[ \text{Controller parameters} \]

where \( p \) and \( c \) do not affect the controller ability to reject disturbances in the output, only the ability to track setpoint changes.

\[ \text{Controller parameters} \]

in the Laplace domain we get

\[ \text{Controller parameters} \]

If the parameters of the series form are known, the corresponding parameters of the parallel form can be calculated according to

\[ \text{Controller parameters} \]

The condition for \( \delta \) is as in (7.6).

\[ \text{Controller parameters} \]

A simple way of obtaining 2DOF PID controller is to use setpoint weighting. With the definitions

\[ \text{Controller parameters} \]

where \( p \) and \( c \) are setpoint weights, the control law becomes

\[ \text{Controller parameters} \]

A simple way of obtaining 2DOF controller is to use setpoint weighting. With the definitions

\[ \text{Controller parameters} \]

\[ \text{Controller parameters} \]

which is a combination of a PI controller and a PD controller.

\[ \text{Controller parameters} \]

In the Laplace domain we get

\[ \text{Controller parameters} \]

A remedy to this is to include setpoint weighting in the series form of a PID controller.

\[ \text{Controller parameters} \]

In the Laplace domain the control law with setpoint weighting is

\[ \text{Controller parameters} \]

\[ \text{Controller parameters} \]

\[ \text{Controller parameters} \]

\[ \text{Controller parameters} \]

Laboratory Process Control
On-off controller is the simplest type of controller, where the control signal has only two levels. If the variables are defined such that a positive

\[ y(t) = \begin{cases} 1 & \text{if } e(t) > 0 \\ 0 & \text{if } e(t) \leq 0 \end{cases} \]

where \( e(t) \) is the control error. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{P control} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{PI control} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{PD control} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{PID control} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Dead zone} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Integral term} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Gain} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Time delay} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

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\[ \text{PI controller} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{PD controller} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{PID controller} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Controller gain} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Proportional part} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Integral part} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.

\[ \text{Derivative part} \]

The on-off controller is inexpensive, but it causes oscillations in the process. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated. It is often used for temperature control in simple appliances such as ovens, irons, refrigerators and freezers, where oscillations are tolerated.
The ideal form of a PD controller implements the control law
\[ u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau \]
where the gain \( K_p \) and the derivative time \( K_i \) are adjustable parameters; \( e(t) \) is chosen as for a PI controller.

A PD controller is preferred when integral action is not needed, but the dynamics of the process are so slow that the predictive nature of derivative action is useful. Many thermal processes, where energy is stored with small heat losses (e.g., ovens), usually have slow dynamics, almost as integrating systems. A PD controller might then be suitable for temperature control.

Another typical application for PD control is in servo mechanisms such as electrical motors, which usually behave as second-order integrating systems.

As has been shown in Section 7.1, there are many variants of PID controllers. The ideal form and the classical series form have 3 adjustable parameters in addition to \( e(t) \): the proportional gain, the integral time, and the derivative time. If a derivative filter is included, there are 4 adjustable parameters, but the filter time constant is usually selected as a given fraction (e.g., 10\%) of the derivative time.

In addition, the setpoint can be weighted in the proportional part and the derivative part.

The task of a controller is to control a system to behave in a desired way despite unknown disturbances and an inaccurately known system. The controlled system should satisfy performance criteria such as:

- The controlled system must be stable; this is absolutely necessary.
- The effect of disturbances on the controlled output is minimized.
- The controlled output should follow setpoint changes fast and smoothly; this is especially important for setpoint tracking.
- The control error is minimized or kept within certain limits.
- The control signal variations should be moderate or at least not be excessively large; more variations wear out the control equipment faster.
- The control system should be robust (insensitive) against moderate changes in system properties, which introduce model uncertainty.
- The importance of these criteria varies from case to case. Since many criteria are conflicting, compromises have to be made in the control design.

One reason for the fact that there are usually good solutions to the conflicting control criteria is that feedback control is used. However, feedback also introduces limitations because a control error is required for the controller to take action. The closed-loop system thus has a feedback error.

The closed-loop system is then driven to a new steady-state point following a control action.

Another typical application for PD control is in servo mechanisms such as electrical motors, which usually behave as second-order integrating systems.

\[ \frac{\text{Output}}{\text{Input}} = \frac{1}{1 + \tau s} \]

When the gain \( K \) and the derivative time \( T_d \) are adjustable parameters,
\[ u(t) = K (e(t) + \frac{1}{T_d} \frac{de(t)}{dt}) \]

The ideal form of a PD controller implements this control law.
The process dynamics is often the performance limiting factor. Such factors are time delays as well as RHP (right-half plane) poles and zeros in high-order dynamics. In practice, all processes are nonlinear. Such a process cannot be described accurately at different operating points by a linear model with constant parameters; thus there is model/process uncertainty.

Disturbances such as load disturbances and measurement noise limit how well a variable can be controlled. Efficient control of load disturbances often require derivative action, but measurement noise is bad for the derivative. Large load disturbances can cause the control variable to reach its (physical) maximum or minimum value. This is especially troublesome if the controller contains an integrator. Proportional band and integrator windup are two concepts that deal with this limitation.

A controller's proportional band (PB) denotes the maximum control error the controller can handle with the available control signal. The PB is defined for a P controller, but it can be extended to a full PID controller. If the control signal is limited by $\theta_{\text{lim}}$, a P controller can according to (7.24) handle a control error that satisfies

$$\frac{N_{\text{PB}}}{\theta_{\text{lim}} - \theta_{\text{PB}}} \leq \%$$

The PB is equal to $\theta_{\text{lim}} - \theta_{\text{PB}} = \theta_{\text{lim}} - \theta_{\text{PB}}$, where $\theta_{\text{lim}}$ is the highest output ($\theta_{\text{lim}} = \theta_{\text{PB}}$) and $\theta_{\text{PB}}$ is the lowest output ($\theta_{\text{PB}} = \theta_{\text{lim}}$). The PB is then expressed in percent of the total measurable output interval

$$
\displaystyle \theta_{\text{PB}} = \frac{\theta_{\text{lim}} - \theta_{\text{PB}}}{\theta_{\text{lim}} - \theta_{\text{PB}}} \times 100\% = \%$$

If the proportional band is known, the controller gain is given by

$$K_{\text{p}} = \frac{1}{\theta_{\text{lim}} - \theta_{\text{PB}}}$$

In (old) automation systems, the signals are often expressed as a fraction or percentage of the total signal interval (0-1 or 0-100%). The PB is then expressed in percent of the total signal interval. Note that the gain $K_{\text{p}}$ is expressed in terms of the normalized signals, which means that the controller gain is dimensionless. Usually controllers are tuned for stability and performance, not for signal uncertainty.

Consider the figure, where the PI control law (7.25) has been used. A strong disturbance causes the process output to fall well below the set-point. The controller is not able to eliminate the positive control error, which results in an increase of the integral in the controller. If the disturbance later disappears, the controller will still keep the control error, which can be detrimental to the control performance unless the situation is handled properly.
The described phenomenon is called integral windup (also reset windup).

There are sophisticated as well as simple methods for handling the problem. The term anti-windup is used for such arrangements.

A simple solution is to stop integrating when a control signal reaches a constraint. This requires that:

- It is known when the control signal reaches a constraint (e.g., through measurement)
- There is some built-in logic to interrupt the integration

In the case of digital control, which nowadays is customary, automatic anti-windup can be built into the control law.

Above, some general performance criteria and fundamental limitations to achievable control performance have been considered. Here, some ways of making more specific design specifications will be introduced.

If a process model is available, the specifications make it possible to calculate controller parameters.

It is often desired that the closed-loop response to a step change in the setpoint resembles an underdamped second-order system. Therefore, parameters familiar from the step-response of such a system can be used to specify the desired behavior. Such parameters are:

- The maximum relative overshoot
- The rise time
- The settling time
- The relative damping
- The ratio between successive relative overshoots (or undershoots)

According to the relationships in Section 5.3.3:

The two parameters $\zeta$ and $\omega_n$ are sufficient to determine the transfer function of an underdamped second-order system with a given gain.

The settling time can be used instead of $\zeta$ or $\omega_n$, but the relationships are then only approximate.

The relative damping or the overshoot ratio can be specified instead of $\zeta$.

Some classical tuning recommendations are based on the specification $\zeta = 0.7$, $\omega_n = \sqrt{2}$, and $\tau_s = \frac{1}{4\sqrt{2}}$. This corresponds to an underdamped second-order system with a maximum relative overshoot of 42% and a rise time of 2.2 times the settling time.

For second-order systems, the following relationships hold:

$$\zeta^2 \omega_n^2 = \frac{1}{\tau_s^2}$$


Error specifications

- $\zeta$ (relative damping)
- $\omega_n$ (natural frequency)
- $\tau_s$ (settling time)
- $\tau_r$ (rise time)
- $M_{	ext{max}}$ (maximum relative overshoot)

Some classic tuning rules are based on these error specifications. A commonly used rule is to choose $\zeta$ and $\omega_n$ such that $\tau_s = \tau_r = \frac{1}{4\sqrt{2}}$ and $M_{	ext{max}} = 42%$. These rules are only approximate.

In the case of digital control, with more sophisticated control systems, additional specifications can be used to further improve the performance of the control system.

Above, some general performance criteria and fundamental limitations to achievable control performance have been considered. Here, some ways of making more specific design specifications will be introduced.

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In the case of digital control, with more sophisticated control systems, additional specifications can be used to further improve the performance of the control system.
It is of interest to consider how the error integrals relate to step-response specifications when the controlled system is of second order, i.e.,

\[ Q = \frac{G}{D} \]

In the figure, IAE and ISE are normalized with \( \eta \), ITAE and ITSE with \( \eta^2 \).

As can be seen, every normalized error integral has a minimum for a given relative damping. This damping as well as the corresponding relative over-shoot are shown below.

### Table 7.1 Optimal relative damping for 2nd order system.

<table>
<thead>
<tr>
<th>Error integral</th>
<th>Damping (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IAE</td>
<td>0.66</td>
</tr>
<tr>
<td>ISE</td>
<td>0.59</td>
</tr>
<tr>
<td>ITAE</td>
<td>0.75</td>
</tr>
<tr>
<td>ITSE</td>
<td>0.59</td>
</tr>
</tbody>
</table>

An ideal PID controller of interactive form can be tuned experimentally by making closed-loop control experiments with the real process. The standard feedback structure is used.

1. A P controller \( (Q = d) \) is used for the first experiment. A low value is chosen for the gain \( d \). Note that \( d \) must have the same sign as \( G \).

2. A change in the setpoint \( z \) is introduced. (Some other disturbance could also be used.) The controller gain \( d \) is increased until the output \( y \) starts to oscillate with a constant amplitude (see next slide).

3. The value of the controller gain yielding constant oscillations is denoted \( d \). The period of the oscillations is denoted \( T \).

4. The controller gain is reduced to \( d = \frac{1}{2} \). If the intention was to tune a controller with integral action (PI or PID), an experiment is done with a PI controller using \( d = \frac{1}{2} \).

5. A change in the setpoint \( z \) is introduced. If the output \( y \) is constant, the integral time \( T \) is reduced until the output \( y \) starts to oscillate with a constant amplitude. This occurs when \( T = \frac{1}{2} \).

6. To tune a controller with integral action (PI or PID) an experiment is done with a PI controller using \( d = \frac{1}{2} \).

7. The integral time for a PI or PID controller is chosen as \( T = \frac{1}{2} \).

8. To tune the derivative part of a PID (or PD) controller, an experiment is done with such a controller using \( d = \frac{1}{2} \) and \( T = \frac{1}{2} \). The derivative time is initially set at \( T = \frac{1}{2} \).

9. A change in the setpoint \( z \) is introduced. If the output \( y \) starts to oscillate with a constant amplitude, this occurs when \( T = \frac{1}{2} \).

10. The derivative time for a PD or PID controller is set at \( T = \frac{1}{2} \) for some other disturbance. If the output \( y \) starts to oscillate, the derivative time is reduced until the output \( y \) starts to oscillate with a constant amplitude. This occurs when \( T = \frac{1}{2} \).

If the control performance obtained by the above tuning steps is judged to be unsatisfactory, the controller parameters can be adjusted by "trial and error." The next figure shows how changes of the controller gain \( d \) and the integral time \( T \) typically affect the control performance. The optimal performance is in this case obtained by \( d = \frac{1}{2} \) and \( T = \frac{1}{2} \).
The step response of a purely integrating system is a ramp that changes linearly with time, i.e., it has a constant slope. Any point on the ramp can then be used as a pair of coordinates for calculation of \( \tau \).

Another useful parameter is \( \tau' \) expressed from the point, where the tangent is the equivalent time constant of the system and \( \tau' = \frac{T}{\tau} \).

The step response is a ramp that changes monotonously with time. It is required to determine the characteristic parameters from a step response. A drawback with generating the frequency response is that it is cumbersome and time-consuming to generate oscillations with constant amplitude by adjusting a controller parameter. The critical frequency is the process gain. The critical frequency is the process gain.

The step response to a unit-step response is the process gain. The critical frequency is the process gain.

The tuning correlations are primarily intended for regulatory control; for disturbance rejection. For setpoint tracking, setpoint weighting is suggested. In 2006, Åström and Hägglund showed that, in general, the critical frequency is the process gain.

The step response to a unit-step response is the process gain.
In 1942, Ziegler and Nichols also suggested tunings for P, PI and PID controllers based on the information that can be obtained from a step test. Their recommendations for an ideal controller are given in Table 7.4.

The method requires $\omega_n \geq 1$ and preferably $\omega_n \geq 2$. The CHR tunings (even the aggressive one) are less aggressive than the ZN tuning.

### Table 7.4. Ziegler-Nichols’s controller tuning recommendations based on step response.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$K_c$</th>
<th>$I_T$</th>
<th>$D_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1.0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>PI</td>
<td>0.9</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td>PID</td>
<td>1.2</td>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Note that Ziegler-Nichols’s recommendations based on frequency response and step response do not necessarily give the same controller tuning for the same process.

In 1952, Chien, Hrones and Reswick suggested improvements to Ziegler's and Nichols's recommendations based on a step response. The CHR-method gives different tunings for regulatory control and setpoint tracking tunings for aggressive control (with $\text{overshoot}$) and cautious control (no overshoot).

The method requires $\omega_n \geq 1$ and preferably $\omega_n \geq 2$. The CHR tunings (even the aggressive one) are less aggressive than the ZN tuning.

Note that the different tunings for regulatory control and setpoint tracking can directly be used in a 2DOF controller.

### Table 7.5. Controller tuning for regulatory control by the CHR method.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$K_c$</th>
<th>$I_T$</th>
<th>$D_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.3</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>PI</td>
<td>0.6</td>
<td>4.0</td>
<td>–</td>
</tr>
<tr>
<td>PID</td>
<td>0.95</td>
<td>2.4</td>
<td>0.42</td>
</tr>
</tbody>
</table>

### Table 7.6. Controller tuning for setpoint tracking by the CHR method.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$K_c$</th>
<th>$I_T$</th>
<th>$D_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.3</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>PI</td>
<td>0.35</td>
<td>1.2</td>
<td>–</td>
</tr>
<tr>
<td>PID</td>
<td>0.6</td>
<td>1.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

In 2006, Åström and Hägglund presented improved controller tunings based on a step response. In addition to $\omega_n$ and $\theta$, they use $\theta_d$ in their correlations, which can be used for all $\theta_d < 2$. However, for $\theta_d > 0.4$, they use $\theta_d$ in their tuning based on a step response in addition to $\omega_n$ and $\theta$. The tunings are primarily intended for regulatory control. For setpoint tracking, setpoint weighting can be used as follows:

- **PI control:** $\omega_d = 0$ if $\theta_d < 2$, preferably $\omega_d > 0.1$.
- **PID control:** $\omega_d = 0$ if $\theta_d < 2$, preferably $\omega_d > 0.1$.

### Table 7.7. Åström's and Hägglund's controller tuning correlations.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$K_c$</th>
<th>$I_T$</th>
<th>$D_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.3</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>PI</td>
<td>0.6</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>PID</td>
<td>0.45</td>
<td>0.2</td>
<td>–</td>
</tr>
</tbody>
</table>

Note that there are recommendations for aggressive control for the same controller.
The controller tuning methods in Sections 7.4 and 7.5 employ parameters that can be determined from an experiment with an existing process. If a process model is known, the same parameters can be determined through a simulation experiment, possibly by direct calculation from the process model. For example, a first-order system with a time delay has the transfer function

\[ G(s) = \frac{e^{-\theta s}}{s + 1} \]  

(7.36)

from which the parameters \( \theta \) and \( \tau \) can be calculated according to

\[ \theta = \frac{1}{1 + \tau} \]  

(7.37)

The same tuning methods as in Sections 7.4 and 7.5 can then be used. However, the methods in Sections 7.4 and 7.5 are "general purpose" methods that are not optimized for any specific model type. For a given model, better controller tunings probably exist.

Controller tunings that minimize IAE and ITAE when \( \theta \geq 1 \) are given in Table 7.8 and Table 7.9. These tunings do not guarantee robustness, and the control performance can be bad if the model contains errors.

Cvejn (2009) has derived controller tunings that have a certain robustness, even for systems with large time delays, i.e., for large \( \theta \) values. Table 7.10. Cvejn's tunings for regulatory control and setpoint tracking.

The PI controller tunings tend to give better robustness than the PID controller tunings, which tend to give better performance. We shall consider second-order systems with a time delay but no zeros.

For a given model, better controller tunings probably exist. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used. The same methods as in Sections 4.1 and 5.2 can then be used.
For an overdamped (or critically damped) second-order system, the transfer function is
\[ \frac{Y_s}{N_s}(s) = \frac{1}{\tau_s^2 + \frac{1}{\tau_s} s + 1} \]

The suggested tuning is
\[ \frac{1}{\tau_s} = 2 \]
A strictly proper first-order system without a time delay has the transfer function:

\[ G(s) = \frac{1}{s + \zeta \omega_n} \]

Assume that we want the controlled system to behave as a first-order system with the time constant \( \tau \).

Then, \( G(s) \) becomes:

\[ \frac{1}{s + \frac{\tau}{\tau}} \]

which gives:

\[ G(s) = \frac{\tau}{s + \frac{\tau}{\tau}} \]

Substitution of (7.53) and (7.54) into (7.52) gives:

\[ G(s) = \frac{1}{s + \frac{\tau}{\tau}} \]

which is a PI controller with the parameters:

\[ K_p = \frac{1}{\tau}, \quad \tau_i = \tau \]

Here, \( \tau \) is a design parameter, by which the performance of the control system can be affected.
A second-order system with real poles and a right half plane (RHP) zero has the transfer function
\[
\frac{s^2 + \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}\]
which for (7.76) (gives 1. This corresponds to
\[
\frac{s^2 + \omega_n^2}{s^2 + \omega_n^2} = 1
\]
Let \( G \) have real poles at \(-\omega_n, 0\) and \( s = \omega_n \). This corresponds to
\[
\frac{s^2 + \omega_n^2}{s^2 + \omega_n^2} = 1
\]

The choice of design parameters can be simplified in the following two ways.

1. \( G \) has two equal but real poles at \(-\omega_n, 0\). This corresponds to

2. \( G \) has real poles at \(-\omega_n, -\omega_m\). This corresponds to

The closed-loop transfer function is chosen as
\[
\frac{G(s)}{1 + G(s)F(s)}
\]
Substitution of (7.71) and (7.72) into (7.52) gives
\[
\frac{G(s)}{1 + G(s)F(s)} = \frac{s^2 + 2\zeta \omega_n s + \omega_n^2}{s^2 + \omega_n^2}
\]
which is a PID controller with the parameters
\[
\begin{align*}
K_P &= \frac{\omega_n^2}{\omega_n^2}, \\
K_I &= \frac{2\zeta \omega_n}{\omega_n^2}, \\
K_D &= \frac{\zeta \omega_n}{\omega_n^2}
\end{align*}
\]

In this section, the latter approach is used.
Process Control Laboratory

To illustrate how systems with a time delay can be handled by direct synthesis, a first-order system with a time delay will be studied. Such a system has the transfer function:

\[ G(s) = \frac{1}{1+\tau s} \]  

(7.79)

Calculation of a controller by (7.52) will then result in a controller containing a time delay — there is no practical way to avoid this by the choice of \( K \).

There are methods to implement a controller resulting from (7.52) (see Section 7.8), but not by a regular PID controller. If a PID controller is desired, the time delay has to be approximated in some way.

A standard way of approximating a time delay is to use a Padé approximation. A first-order Padé approximation gives the model:

\[ G(s) = \frac{\frac{1}{\tau s+1}}{\frac{1}{s}+\frac{1}{\tau s+1}} \]  

(7.80)

A natural choice for \( K \) is then:

\[ K = \frac{1}{\tau} \]  

(7.81)

Substitution of (7.81) and (7.84) into (7.52) gives a PID controller with the parameters:

\[ K_p = \frac{1}{\tau (s+1)} \]

(7.82)

Unfortunately, this controller cannot be implemented by a PID controller in a regular feedback loop. In order to do that, the time delay in (7.85) has to be approximated by a rational expression.

If the approximation (7.80) is used, the controller parameters will be:

\[ K = 1, \quad \tau = \frac{1}{s+1} \]

(7.83)

If \( \tau \) is chosen to be as small as possible, the system is unstable. This controller cannot be implemented by a PID controller.

If \( \tau \) is too large, the system is stable. However, the controller gain must then be very small, which is undesirable.

Internal model control (IMC) is closely related to direct synthesis (DS). As in DS, a model of the system to be controlled is explicitly built into the controller, but in a different way.

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not inverted.

The choice of $G$ to be inverted: non-minimum phase parts of $G$ are determined by the stability and realizability of $G$. If the choice of $G$ is handled in a feedback with a derivative filter, then a stable IMC controller may not contain integral action.

A feedback closed-loop system $G$ requires a stable IMC controller.

The following conditions can be drawn from (7.93):  

The IMC controller design.

7.8.4 Controller design

7.8 Internal model control

---

If the filter is chosen as $G_f = G_f$, this is the same as (7.25).  

\[
\frac{G(s) - G_f(s)}{G(s)} = G(s) G_f(s) = G_f(s) \
\]

where $G_f = \frac{1}{s^2}$. Substitution of (7.88) into (7.87) gives

\[
\frac{1 + s^2}{1 + s^2} = \frac{1}{1 + s^2} \
\]

which is a PD controller with a derivative filter having the parameters $\frac{1}{1 + s^2}$.

Consider a system modeled as a first-order system with a time delay $\tau$.

The IMC structure.
### 7.8 Internal model control

#### 7.8.5 Implementation with a regular PID controller

An advantage of the IMC structure is that time delays can be handled exactly, but only a regular PID controller is permitted, because the process model $G_c$ can always be factored as $G_c = G d_0 G_m$, where $G_m$ contains all non-minimum-phase elements of $G_c$, but no minimum-phase elements, and normalized so that $G_m(0) = 1$ (i.e., it contains only RHP zeros and time delay). This means that $G_m$ contains all RHP zeros and time delay, but no non-minimum-phase elements (not even a time delay), and that the full IMC block diagram — the use of $G_m$ is only a technical aid for the calculation of $G_c$.

The IMC filter could be chosen as the desired closed-loop transfer function without any non-minimum-phase elements (not even a time delay), but in practice a low-pass filter

$$G_c = \frac{1}{(s+1)}$$

Note that the full $G_m$ should be used as internal model as illustrated by the IMC block diagrams — the use of $G_m$ is only a technical aid for the calculation of $G_c$.

When $G_m$ is calculated according to (7.38) only $G_m(0) = 1$.

### 7.9 PID Controllers

#### 7.9.1 Controller tuning based on step-response parameters

The desired time constant of the closed-loop system is $T_r$, which is used in the calculations. $T_r$ is used in several expressions.

#### 7.9.2 Controller tuning based on frequency-response parameters

Table 7.12. IMC-based tuning of ideal PID controller.

<table>
<thead>
<tr>
<th>$T_r$</th>
<th>$T_p$</th>
<th>$K_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>0.66</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>0.7</td>
<td>2.1</td>
<td>0.85</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
<td>0.55</td>
</tr>
<tr>
<td>0.8</td>
<td>2.4</td>
<td>0.95</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The process model $G_c$ can always be factored as $G_c = G d_0 G_m$, where $G_m$ contains all non-minimum-phase elements of $G_c$, but no minimum-phase elements, and normalized so that $G_m(0) = 1$ (i.e., it contains only RHP zeros and time delay). This means that $G_m$ contains all RHP zeros and time delay, but no non-minimum-phase elements (not even a time delay), and that the full IMC block diagram — the use of $G_m$ is only a technical aid for the calculation of $G_c$.
iterate (i.e., first guessing).

If a first-order model is desired, the half rule gives

\[ \frac{1 + s \tau}{1 + s \tau} \approx \frac{1 + \frac{s}{\tau}}{1 + \frac{s}{\tau}} \]

Next, the smallest remaining denominator time constant is selected. If there is no such time constant, or if that order is not known, the suggested value is \( \tau \), which is the time delay in the model. Since this is not initially known, one may have to try a number of values for \( \tau \) to find the one currently being considered. It is considered closest to \( \tau \) if it is chosen.

The ratio \( \frac{1 + s \tau}{1 + s \tau} \) is now approximated as

\[ \frac{1 + \frac{s}{\tau}}{1 + \frac{s}{\tau}} \approx \frac{1 + \frac{s}{\tau} + \frac{s}{\tau}}{1 + \frac{s}{\tau} + \frac{s}{\tau}} \]

where \( \tau \) is the desired closed-loop time constant. If \( \tau \) is not known, it is considered closest to \( \tau \) if it is chosen. It is considered if \( \tau \) is not selected. If there is no such constant of order \( \tau \) that

Note that the gain as well as the values and number of denominator time constants may have changed from the original model. Since this is not initially known, one may have to try a number of values for \( \tau \) to find the one currently being considered.

The above procedure gives an approximate minimum-phase part of the original system. The transfer function to be simplified is factorized into a minimum-phase part and a non-minimum-phase part.

Emphasis by suitable pole-zero cancellation. Any left-half plane (LHP) poles of the minimum-phase part have the form

\[ (1 + s \tau)^n (1 + s \tau)^{-n} \]

such that

\[ \frac{1 + \frac{s}{\tau}}{1 + \frac{s}{\tau}} \approx \frac{1 + \frac{s}{\tau} + \frac{s}{\tau}}{1 + \frac{s}{\tau} + \frac{s}{\tau}} \]

where \( \tau \) is the desired closed-loop time constant. If \( \tau \) is not known, it is considered closest to \( \tau \) if it is chosen. It is considered if \( \tau \) is not selected. If there is no such constant of order \( \tau \) that

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Process Control Laboratory

Simplify the model to a second-order model by Skogestad's method and determine the parameters of a PID controller by IMC-based tuning for this model. Use a first-order filter time constant $\tau = 1$. Here $\tau = 1$.

According to (7.108c), $\tau = 1$. The numerator factor $(1+\frac{\tau + \tau_1}{\tau})$ can now be cancelled out against the new denominator factor, which gives $\tau = 1$ and $\tau = 1$

The resulting second-order model is $\frac{\tau + \tau_1}{\tau}$ with $\tau_1 = 1$. According to Table 7.12 for IMC-based tuning of second-order model:

\[
\begin{align*}
\tau_1 & = \tau + \tau_2 - \tau_2 - \tau_1 \\
\tau_2 & = \tau + \tau - \tau_1 \\
\end{align*}
\]

Example 7.2. Isaksson and Graebe (1999) have presented a method to simplify a high-order model, where the fast and slow dynamics are combined to yield a model with a desired number of poles and zeros. If the original model contains a time delay, it is either left intact or substituted by a Padé approximation.

To describe the method, both factorized and polynomial forms of the original transfer function are employed. If the numerator order is $\rho$ and the denominator order is $\eta$, the transfer function is

\[
\begin{align*}
\frac{\tau + \tau_1}{\tau} & = \frac{-\tau + \tau_2}{\tau + \tau_2} \frac{\tau + \tau_1}{\tau + \tau_2} \frac{\tau + \tau_1}{\tau + \tau_2} \frac{\tau + \tau_1}{\tau + \tau_2} \frac{\tau + \tau_1}{\tau + \tau_2} \\
\end{align*}
\]

Complex-conjugated poles or zeros is no problem, except if they occur as poles number $\eta$ and $\eta+1$ or zeros number $\rho$ and $\rho+1$. One solution is then to use the real part of the complex conjugate as $\tau_1 = \tau + \tau_2 - \tau_2 - \tau_1$.

If the model is to be used for controller tuning, a strictly proper first- or second-order model, possibly with a time delay, is usually desired. Then $\tau = 1$

\[
\begin{align*}
& (1+\frac{\tau + \tau_1}{\tau}) \frac{\tau + \tau_1}{\tau} = (s) G \\
& (1+\frac{\tau + \tau_1}{\tau}) \frac{\tau + \tau_1}{\tau} = (s) G \\
& \text{As allowed at Exam} \\
\end{align*}
\]
time-varying disturbances can often be approximated by sinusoidal signals. The study of sinusoidal inputs is useful because measurement noise and amplitude and frequency of the input sinusoids. The properties of certain characteristic properties that depend on the system as well as the system is excited by a sinusoidal input. The model gives a sinusoidal input for the frequency. Stationary behavior refers to the situation when initial effects have vanished, i.e., when the time function. In this chapter, system properties are studied in the frequency domain (e.g., stability) and time domain (e.g., step response). Table 7.12 for IMC-based tuning of second-order model then gives the model equations. From Table 7.12, for a cascaded singular of second-order model then gives

\[
\frac{Y(s)}{N(s)} = \frac{1+q+q^2}{1+(1+q+q^2)(1+q+q^2)} = \frac{(5)q}{q^2+2q+1}
\]

From Example 2.3, the model reduction method used is Example 2.2 by literature and graphs. Example 7.2: Use the model reduction method.
is a phase-advancing element.

Thus, the negative system gain causes a phase shift of \( \pi \) radians, or \( 180^\circ \), in the output.

It is of interest to rewrite (8.2) so that the phase shift is explicitly seen in the equation:

\[
\phi = \phi_{\text{output}} - \phi_{\text{input}}
\]

For the definition of the frequency response of higher-order systems, dynamics play a prominent role.

The frequency response for an arbitrary linear, stable, and stationary output of any stable linear system with an input (8.2) can be written as:

\[
\mathcal{H}(s) = \mathcal{H}(\omega) = |\mathcal{H}(\omega)| e^{j\phi(\omega)}
\]

A system with the transfer function (8.4) is an output, and \( \mathcal{H}(\omega) \) is the system gain. Therefore, the frequency response is a multiplier of the sinusoidal input.

The ratio between the amplitudes of the output and input signals is also of interest. In this case, the \( \mathcal{H}(\omega) \) is not the amplitude ratio and \( \phi(\omega) \) is the phase shift:

\[
|\mathcal{H}(\omega)| \cdot e^{j\phi(\omega)}
\]

In general, \( |\mathcal{H}(\omega)| \) is the amplitude ratio and \( \phi(\omega) \) is the phase shift of the system:

\[
|\mathcal{H}(\omega)| = |\mathcal{H}(\omega)| e^{j\phi(\omega)} = \mathcal{H}(\omega)
\]

The phase shift is then:

\[
\phi(\omega) = \phi_{\text{output}} - \phi_{\text{input}}
\]

There are now two situations regarding the phase of the output:

1. If \( \Re(\mathcal{H}(\omega)) > 0 \) and \( \phi(\omega) < 0 \), the phase of the output is before the phase of the input. Thus, the output can be written as:

\[
|\mathcal{H}(\omega)| \cdot e^{j\phi(\omega)} = |\mathcal{H}(\omega)| e^{j\phi(\omega)}
\]

2. If \( \Re(\mathcal{H}(\omega)) < 0 \) and \( \phi(\omega) > 0 \), the phase of the output is after the phase of the input. Thus, the output can be written as:

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\]

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\[
|\mathcal{H}(\omega)| \cdot e^{j\phi(\omega)} = \mathcal{H}(\omega)
\]
If the gain $\zeta > 1$, this has to be taken into account as for a static system.

Thus, the output is given by (8.7) with

$$s \omega_0 (s) = \phi$$

Parallel connection of static and derivative system

A system composed of a parallel connection of a static and a derivative system has the transfer function

$$G(s) = \frac{1}{\zeta (s + 1)}$$

This can, e.g., be the transfer function of a PD controller. From (8.3) and (8.25)

$$s \omega_0 (s) = \phi$$

The sinusoidal input (8.2) gives

$$s \omega_0 (s) = \phi$$

The output is thus given by (8.7) with

$$s \omega_0 (s) = \phi$$

Parallel connection of static and integrating system

A system composed of a parallel connection of a static and an integrating system has the transfer function

$$G(s) = \frac{1}{\zeta s^2 + 1}$$

This can, e.g., be the transfer function of a PI controller. From (8.3) and (8.25)

$$s \omega_0 (s) = \phi$$

The sinusoidal input (8.2) gives

$$s \omega_0 (s) = \phi$$

The output is thus given by (8.7) with

$$s \omega_0 (s) = \phi$$
A time delay of length $\tau$ has the transfer function $G(s) = e^{-\tau s}$. In the time domain, the relationship between the output and the input is

$$\text{output} = \text{input} \cdot e^{-\tau \text{time}}.$$

A first-order system is described by the differential equation

$$\frac{d}{dt} y(t) + \tau \frac{d^2}{dt^2} y(t) = \tau \frac{d^2}{dt^2} x(t),$$

where $\tau$ is the time constant of the system.

The Laplace transformation of (8.33) yields

$$Y(s) = G(s)X(s),$$

where $G(s)$ is the transfer function of the system. The Laplace transforms of the sinusoidal inputs (8.2) are

$$X(s) = \omega \sin \omega t, \quad Y(s) = \omega \sin \omega t,$$

The Laplace transform of the exponential input is

$$X(s) = \frac{1}{s^2 + \omega^2}, \quad Y(s) = \frac{\omega}{s^2 + \omega^2}.$$
The identity (8.40) has to apply for arbitrary values of \( \varepsilon \).

Choosing \( \varepsilon = \omega \scriptscriptstyle \nu \) means that

\[
\varepsilon + \omega \scriptscriptstyle \nu = 0.
\]

Then (8.40) yields

\[
\omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \varepsilon \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.41)

The identity (8.41) requires that the real part and the imaginary part are satisfied independently. Because

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \varepsilon \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu
\]

(8.42)

The first-order system (8.34) yields

\[
\varepsilon \omega \scriptscriptstyle \nu = \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.43)

From which

\[
\alpha \omega \scriptscriptstyle \nu = \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.44)

Substitution of (8.44) into (8.42) and further into (8.39) yields the stationary solution

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.45)

The trigonometrical identity (8.16) applied to (8.45) yields

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.46)

Thus, the stationary solution has the same form as (8.7), i.e.,

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.47)

For a first-order system (8.47) applies with

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.48)

A second-order system without zeros and with no time delay has the transfer function

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.49)

Analogously to (8.36), the Laplace transform of the sinusoidal input yields

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.50)

When \( \omega \) < \( \omega \omega \nu \), there always exists a partial fraction expansion

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.51)

According to our Laplace transform table, the first term on the right-hand side has an inverse transform containing the factor

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.52)

When \( \omega \) < \( \omega \omega \nu \), there always exists a partial fraction expansion

\[
\varepsilon \omega \scriptscriptstyle \nu = \varepsilon \omega \scriptscriptstyle \nu + \alpha \omega \scriptscriptstyle \nu \omega \scriptscriptstyle \nu = 0.
\]

(8.53)

Therefore, the stationary solution to (8.53) is given by (8.39) and the coefficients \( \omega \) and \( \varepsilon \) are given by (8.42).
According to the theory of complex number, magnitude of \( (\omega) \) is a complex number, it can be characterized by its process of magnitude and phase shift. According to the theory of complex number, magnitude of \( (\omega) \) is a complex number, it can be characterized by its process of magnitude and phase shift.

To simplify the notation, we define
\[
\phi = \frac{A_2(x)}{A_0(x)} = \frac{\sin(A_2(x))}{\sin(A_0(x))} = \phi(x)
\]

From (8.42) and further into (8.39) gives
\[
\phi(x) = \frac{B(x)}{A(x)} = \frac{1}{\sin(x)}
\]

Substitution into (8.42) and further into (8.39) gives
\[
\phi(x) = \frac{B(x)}{A(x)} = \frac{1}{\sin(x)}
\]

In this case
\[
\phi(x) = \frac{B(x)}{A(x)} = \frac{1}{\sin(x)}
\]
This ambiguity can be solved by adding up the phase shifts of the individual subsystems as given by (8.72). If the system has complex-conjugated poles or zeros, they can be decomposed into the series-connected subsystems (8.79a) and (8.79b) without any detectable difference, because we know their amplitudes (or magnitudes) of the subsystems and amplitudes (or magnitudes) of the full system is obtained as the product of the phases. Therefore, the transfer function (8.78) and (8.79) mean that for the full system the Eqs. (8.78) and (8.79) can be expressed as

\[
\begin{align*}
\mathcal{V} & = \mathcal{Y} \\
\mathcal{V} & = \mathcal{I} \\
\mathcal{V} & = \mathcal{X} \\
\mathcal{V} & = \mathcal{Z} \\
\mathcal{V} & = \mathcal{W} \\
\mathcal{V} & = \mathcal{V}
\end{align*}
\]

Substitution into (8.78) yields

\[
(\omega_0 - \xi)^2 = \omega_0^2
\]

However, because of the properties of the exponential function, this ambiguity can be avoided.

By the method in Section 5.2.1, this ambiguity can also be avoided.

In terms of Bode plots, it is important to note that the phase angles of complex numbers can be expressed

\[
\arg(\mathcal{V}(\omega)) = \arg(\mathcal{I}(\omega)) + \arg(\mathcal{Y}(\omega)) + \arg(\mathcal{Z}(\omega)) + \arg(\mathcal{W}(\omega))
\]

From (8.79) it follows that the frequency response of the full system is

\[
\begin{align*}
\mathcal{V} & = \mathcal{Y} \\
\mathcal{V} & = \mathcal{I} \\
\mathcal{V} & = \mathcal{X} \\
\mathcal{V} & = \mathcal{Z} \\
\mathcal{V} & = \mathcal{W} \\
\mathcal{V} & = \mathcal{V}
\end{align*}
\]
This gives several possibilities of representing the Bode diagram graphically, e.g.,

\[ \log |G(j\omega)| = \text{phase shift} \]

The phase shift and the magnitude are calculated as the frequency response of the system with the transfer function \( G(s) \) of a system with the transfer function \( G(s) \) is calculated in the frequency domain.

Since \( G(j\omega) \) is a complex number, it can be represented in two ways:

1. \( G(j\omega) \) can be considered as a system property.
2. The system—except for a multiple of \( 2\pi \) in the phase shift—the frequency response can be combined. Information about the frequency response of \( G(s) \) contains information about the frequency response of \( G(s) \) is obtained in the Bode diagram, but in the complex plane.

The frequency is expressed on a logarithmic scale in both diagrams.

In the course, degrees are used instead of radians, whereas either the degree scale (deg) or the sine scale (rad) is plotted on a logarithmic scale.

The absolute value \( |G(j\omega)| \) is plotted on a logarithmic scale, either.

The frequency to two diagrams is expressed as a pure amplitude ratio or with the logarithmic "unit" dB, defined as

\[ (8.81) \]

Table 8.1: Frequency response of low-order systems.

 Allowed at Exam 8.2 Graphical frequency response

8.2.1 Development

8.2.2 Graphical frequency response analysis
The transfer function of a second-order system is

\[ \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

The Bode diagram displays the amplitude ratio:

\[ A_m(\omega) = \frac{1}{\sqrt{1 + (2\zeta \omega / \omega_n)^2}} \]

The phase shift:

\[ \phi(\omega) = -\tan^{-1}(2\zeta \omega / \omega_n) \]

At high frequency, the slope of the amplitude plot is

\[ \frac{\text{dB}}{\text{dec}} = \frac{20 \log_{10}(\omega / \omega_n)}{\text{dec}} = \frac{20 \log_{10}(\omega)}{\omega_n} \]

The diagram applies to all second-order systems due to normalized axes.
oscillation will become larger and larger and the system is unstable.

If the system in the block diagram is unstable, the system is unstable.

The system in the block diagram is unstable.

The system in the block diagram is unstable.

The system in the block diagram is unstable.

The system in the block diagram is unstable.

The system in the block diagram is unstable.
The smallest frequency such that the loop transfer function \( \zeta \triangleq \frac{1 \pm \omega}{\omega} \) has a phase shift equal to \(-\pi/2\) is determined. This frequency, \( \omega_c \), is the critical frequency of the system.

The amplitude ratio of the loop transfer function at this frequency, i.e., \( \frac{\zeta}{\omega_c} \), is determined.

The closed-loop system is stable, if \( \frac{\zeta}{\omega_c} \) is less than 1, unstable, if \( \frac{\zeta}{\omega_c} \) is greater than 1.

For the loop transfer function \( G(s) \), the critical frequency \( \omega_c \) can be determined by solving the equation \( \zeta(\omega_c) = \pm 1 \). Numerically, this can be solved iteratively to find \( \omega_c \).

A process that can be modeled as a pure time delay is controlled by a P controller, the controller gain \( K_c \) is determined using the Bode diagram of the loop transfer function.

When a small change in the setpoint is made, the controlled process starts to oscillate with constant amplitude and the period 10 min. which is equal to \( \frac{2\pi}{10} \) rad/min.

\( R = \) allowed at Exam
Example 8.32 Stability margins

(a) We need to find $\omega_c$, such that

$$\omega_c = \frac{\phi}{2}$$

From which

$$\omega_c = 0.5 \text{ rad/min}$$

(b) To check if the system is stable with

$$\omega_c = 0.5 \text{ rad/min}$$

using, e.g.,

$$\omega_c = 0.5 \text{ rad/min}$$

This converges to

$$\omega_c = 2 \text{ rad/min}$$

This is obtained if

$$\omega_c = 2 \text{ rad/min}$$

Thus, this is the critical frequency

$$\omega_c = 2 \text{ rad/min}$$

Time delay changes to

$$\omega_c = 2 \text{ rad/min}$$

The phase margin is defined

$$\omega_c = 2 \text{ rad/min}$$

Mathematically, the phase margin is defined

$$\omega_c = 2 \text{ rad/min}$$

When the amplitude of the loop transfer function is 1

$$\omega_c = 2 \text{ rad/min}$$

The phase margin is defined as

$$\omega_c = 2 \text{ rad/min}$$

The phase margin is defined as

$$\omega_c = 2 \text{ rad/min}$$

Note that the phase curve is independent of gains.

Thus,

$$\omega_c = 2 \text{ rad/min}$$

Starting value.

The new gain margin is found by

$$\omega_c = 2 \text{ rad/min}$$

This converges to

$$\omega_c = 2 \text{ rad/min}$$

From which

$$\omega_c = 2 \text{ rad/min}$$

The gain margin is defined as the ratio of the gain of the loop transfer function with

$$\omega_c = 2 \text{ rad/min}$$

and

$$\omega_c = 2 \text{ rad/min}$$

This is obtained if

$$\omega_c = 2 \text{ rad/min}$$

Then the solution is obtained numerically. The phase shift equation is

$$\phi = 2 \text{ rad/min}$$

Example 8.32 Stability margins

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$$\phi$$

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Mathematically, the phase margin is defined

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When the amplitude of the loop transfer function is 1

$$\phi = 2 \text{ rad/min}$$

The phase margin is defined as

$$\phi = 2 \text{ rad/min}$$

The phase margin is defined as

$$\phi = 2 \text{ rad/min}$$

Note that the phase curve is independent of gains.

Thus,

$$\phi = 2 \text{ rad/min}$$

Starting value.
If divergence occurs, it is in the denominator. Complex-conjugated poles and zeros will also be allowed at Exam if a factor is to be found from the Bode diagram. The phase shift is expressed in radians.

The phase equation is derived:

\begin{align*}
\phi &= \theta - \omega T \\
\phi &= \text{cross-over frequency} - \text{critical frequency}
\end{align*}

The phase shift is expressed in radians. The phase equation is derived:

\begin{align*}
\phi &= \theta - \omega T \\
\phi &= \text{cross-over frequency} - \text{critical frequency}
\end{align*}
depends on the iteration method. A can be calculated iteratively by (8.104)

\[ \left( \frac{\omega_{0}}{\omega} \right) \left( I - \omega T \right) Z + 1 = \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) + 1 \]

The following function is defined:

\[ K = \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) - I \]

The system (8.99) has the amplitude ratio

\[ K = (8.105) \]

\[ \text{The phase equation.} \]

If desired, the iteration can be continued with which often is known, would usually be a good starting value.

According to the Newton-Raphson method, (8.99) \( \text{may also be used.} \)

\[ \frac{3m + \omega}{3m + \omega} = \frac{3(n+1) + 1}{3(n+1) + 1} \]

where \( m \) is the solution of (8.99) and \( n \) is a parameter that

\[ (8.104) \]

\[ (\omega_{0}) \theta \delta - \omega_{0} = 1 + \delta \]

\[ (8.105) \]

\[ \frac{(\omega_{0})^{2} + (\delta)^{2}}{(\omega_{0})^{2} + (\delta)^{2}} \]

\[ \frac{(\omega_{0})^{2} + (\delta)^{2}}{(\omega_{0})^{2} + (\delta)^{2}} \]

By (8.99), \( \theta \delta \) can be calculated iteratively by (8.106)

\[ (8.106) \]

\[ (8.107) \]

\[ \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) + 1 \]

\[ \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) + 1 \]

\[ \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) + 1 \]

\[ \left( \frac{\omega_{0}}{\omega} \right) \left( \frac{\omega_{0}}{\omega} \right) + 1 \]

The corresponding complex-conjugated time constants are \( \tau \approx \omega_{0} \).

A is substituted into (8.103). In (8.106), the substitution

\[ (8.107) \]

\[ (8.108) \]

\[ (8.104) \]

\[ (8.104) \]

\[ (8.105) \]

\[ (8.106) \]

The corresponding complex-conjugated time constants are \( \tau \approx \omega_{0} \).

If there are complex poles or zeros, they occur as complex-conjugated pairs. Such a pair has the corresponding complex-conjugated time constants, which might also be used.

Laboratory

Laboratory
The control design is as follows.

Exercise 8.32

3. The integral time is \( T_i = \frac{5}{\sigma} \)

\[
L = \frac{\sigma}{\sigma - 1}
\]

2. Solve (8.112) for \( K \) with \( \sigma = 0.5 \),

\[
K = \frac{1}{1 + \frac{s}{5}}
\]

1. Solve (8.112) for \( \sigma \) with \( K = 1 \).

\[
\sigma = \frac{5}{s}
\]

The integral time is \( T_i = \frac{5}{\sigma} \) where \( \sigma \) is the cross-over frequency.

Design a controller for a desired phase margin.

\[
\frac{\sigma}{\sigma - 1} + \frac{1}{1 + \frac{s}{5}} = \frac{1}{1 + \frac{s}{5}}
\]

The amplitude ratio and the phase shift of the loop transfer function are

8.4 Frequency response analysis

8.4.4 Design of PI controllers

The cross-over frequency is \( \omega_c \) where \( \sigma \geq 0.5 \).

A PI controller has the transfer function

\[
\frac{1}{1 + \frac{s}{5}}
\]

\[
G(s) = \frac{1}{1 + \frac{s}{5}}
\]

A PI controller design is

8.4.5 Controller design in the frequency domain

8.5 Design of PID controllers

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8.5.5 Design of PID controllers

The used stability criteria are the gain margin and the phase margin to achieve the phase margin. Also calculate the gain margin and the phase margin.

In this section, it is shown how PI and PD controllers can be designed.
The figure shows simulated step responses for Example 8.1.

This means that the maximum phase lift should be enough to satisfy the phase margin requirement, but excessive derivative action is not desired. This means that the maximum phase lift is obtained at the frequency where the filter time constant is the phase margin.

The maximum phase shift is positive and

where the desired cross-over frequency is 

The derivative of the PD controller that produces the phase lift is

 allowed at Exam.

The amplitude ratio and phase shift are defined at the cross-over frequency, where the maximum phase lift is obtained.

The PD controller causes a phase shift

A derivative action is not desired. This means that the maximum phase lift is obtained at the frequency where the derivative action is

The PD controller with a derivative action is not desired.

A derivative action can be described as

A PD controller can be described as a second order system.

The control gain is 

and the integral time is 

The control gain is 

The control gain is 

Because a larger phase margin means a lower cross-over frequency, 

The control gain is 

A derivative action can be described as a second order system.

The control gain is 

A derivative action can be described as a second order system.

The control gain is 

A derivative action can be described as a second order system.

The control gain is 

A derivative action can be described as a second order system.

The control gain is 

A derivative action can be described as a second order system.

The control gain is 

A derivative action can be described as a second order system.
The solution of (8.132) depends on how \( \omega_n \) is specified.

\[
\omega_n = \sqrt{\frac{1}{2m} + \frac{1}{K}}
\]

Solution of (8.132) for the numerator and denominator.

In (8.131), \( \omega_n \) and \( \zeta \) are combined.

\[
0 = \frac{2\zeta \omega_n \xi}{\zeta \omega_n + 1}
\]

\[
I = \frac{2\zeta \omega_n \xi}{\zeta \omega_n + 1} \left( \frac{\omega_n^2 + 1}{\omega_n^2} \right)
\]

Eq. (8.125) gives

\[
\frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2} = \frac{2\zeta}{\omega_n^2}
\]

Eq. (8.124) gives

\[
\frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2} = \frac{2\zeta}{\omega_n^2}
\]

Eq. (8.123) gives

\[
\frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2} = \frac{2\zeta}{\omega_n^2}
\]

Eq. (8.122) gives

\[
\frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2} = \frac{2\zeta}{\omega_n^2}
\]

Eq. (8.121) gives

\[
\frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2} = \frac{2\zeta}{\omega_n^2}
\]

As for the PD controller design, it is reasonable to choose the maximum phase lift of the PD part of the PID controller to occur at the cross-over frequency.

To achieve the phase margin, \( \omega_n \) and the cross-over frequency \( \omega_c \) can be remedied by connecting a PI and a PD controller in series. This can be achieved by combining PI and PD controllers in series. This control action will not be used to achieve the desired integral action. Thus, performance (cross-over frequency) and steady-state errors are not necessary.

A PD controller can be designed in the frequency domain as follows.

1. Calculate the derivative time from (8.122), i.e.,

\[
T_d = \frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2}
\]

2. Calculate the derivative filter time constant from (8.118), i.e.,

\[
T_d = \frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2}
\]

3. Calculate the controller gain from (8.121), i.e.,

\[
K_c = \frac{1}{\omega_n^2 + 1} + \frac{\xi}{\omega_n^2}
\]

A PID controller for desired phase margin and cross-over frequency can now be designed as follows.

8.4.2 Design of PID controllers

8.4.3 Frequency Response Analysis

8.4.4 Controller Design in the Frequency Domain
Eq. (8.133) gives

\[ G(s) = \frac{1 + \frac{1}{\tau s}}{\tau s} \]

To achieve the phase margin \( \phi_m = 60^\circ \) and the cross-over frequency \( \omega_c \) to design a PID controller with a derivative filter for a system with the transfer function.

Example 8.3: Controller design in the frequency domain.

8.4.3 PID controller

8.4. PID controller for desired phase margin and cross-over frequency

1. Calculate the derivative gain from (8.132), i.e.,

\[ \left( \frac{d\omega_c}{d\omega} \right)_G = \frac{\omega_c}{\omega} \]

2. Calculate the derivative gain from (8.132), which can be rewritten as

\[ \omega_c = \frac{G(s)}{1 + \frac{1}{\tau s}} \]

3. Calculate the derivative gain from (8.132), i.e.,

\[ \omega_c = \frac{G(s)}{1 + \frac{1}{\tau s}} \]

8.4.3 PID controller

8.4.3 PID controller

8.4. PID controller for desired phase margin and cross-over frequency
Consider a system with the transfer function.

Example 9.1: State-space model from transfer function.

9.2.2 State-space models

The state of a system represents the minimum information about the system.

The state of a system is the same as its physical representation.

A state variable is defined as the minimum number of variables.

On the basis of (9.4), the state variables can be defined as follows:

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \]

where \( x_1(t), x_2(t), \ldots, x_n(t) \) are the state variables.

For a continuous-time linear system, the relationship between the state variables and the output can be written as a system of first-order differential equations:

\[ \frac{dx}{dt} = Ax \]

where \( A \) is a constant matrix, and \( x \) is the state vector.

The concept of state variables is a natural concept. The state variables are so-called because they describe the state of a system.

A state-space model is based on the concept of state variables and states.

such as:

\[ y(t) = Cx(t) + Du(t) \]

where \( y(t) \) is the output vector, \( u(t) \) is the input vector, \( C \) is the output matrix, and \( D \) is the feedthrough matrix.

Such a model is often called a state-space model.

A more complex and detailed model is described.

For systems with multiple input signals and multiple output signals (MIMO).
Development of the derivatives in (Ex 9.2.2) according to the principle

\[
\begin{align*}
\frac{d}{dt}A &= \frac{d}{dt}A + \frac{d}{dt}A - \frac{d}{dt}A,
\end{align*}
\]

leads to the system matrix

\[
A = (\frac{d}{dt}A) - (\frac{d}{dt}A) = A + (\frac{d}{dt}A) - (\frac{d}{dt}A) = A.
\]

Equations (Ex 9.1.4–5), including the initial conditions, can be expressed as the matrix equation

\[
[\begin{bmatrix} (1) & (2) & (3) \\
(1) & (2) & (3) \\
(1) & (2) & (3) \end{bmatrix}] [\begin{bmatrix} (1) \\
(2) \\
(3) \end{bmatrix}] = [\begin{bmatrix} (1) \\
(2) \\
(3) \end{bmatrix}].
\]

Example 9.1
As shown above, a state-space model can be read as a linearization of a state-space model.

Consider now the relationship

\[
\begin{align*}
(1) \n \eta &= 0 \\
(2) \n \eta &= 0
\end{align*}
\]

where all states and outputs are considered. (9.9) can be written

\[
(\eta \times (\eta \times)) = (\eta \times)
\]

or

\[
(\eta \times (\eta \times)) = (\eta \times)
\]

Consider the nonlinear state-space model

\[
(\text{Ex 9.2.6})
\]

\[
(\text{Ex 9.2.5})
\]

\[
(\text{Ex 9.2.4})
\]

\[
(\text{Ex 9.2.3})
\]

\[
(\text{Ex 9.2.2})
\]

\[
(\text{Ex 9.2.1})
\]

\[
(\text{Example 9.2})
\]

\[
(\text{Example 9.1})
\]

\[
(\text{Example 9.0})
\]
Thus, \( G(s) = \frac{1}{s^2 + 1} \) is the transfer function of the system.

The reason is that two poles are cancelled out at \( s = 1 \).}

\[
\begin{align*}
I = I + \left[ \frac{s}{0} \right]_{0} & = I + \left[ \frac{0}{1} \right]_{0} \\
I = I + \left[ \frac{s}{0} \right]_{0} & = I + \left[ \frac{0}{1} \right]_{0} \\
I = I + \left[ \frac{s}{0} \right]_{0} & = I + \left[ \frac{0}{1} \right]_{0} \\
I = I + \left[ \frac{s}{0} \right]_{0} & = I + \left[ \frac{0}{1} \right]_{0}
\end{align*}
\]

The lower transition of a state-space model is defined at Exam 9.2.1. From state-space model to transfer function.

\[ (s) \begin{cases} nG(s) = (s)A + (s)B(s-V-I)C = (s)A \\ nA + (s)nB_{-1}(s-V-I)+(-s)X_{-1}(s-V-I)C = (s)A \end{cases} \]

where \( n \geq 0 \) is the initial state, \( X \) is the state, \( V \) is the input, \( I \) is the disturbance, and \( C \) is the output.

Equation 9.2.1 yields

\[ (s)A + (s)X = (s)A \]

The lower transition of a state-space model is defined at Exam 9.2.1. From state-space model to transfer function.

\[ (s)A + (s)X = (s)A \]

The lower transition of a state-space model is defined at Exam 9.2.1. From state-space model to transfer function.
Examine the controllability and observability of the system in the Example 9.3.

The transformed system (Example 9.3) is given by:

\[
A' = A + B_1 \begin{pmatrix} V & I \end{pmatrix} C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

The controllability matrix of the transformed system is:

\[
C' = \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

The observability matrix of the transformed system is:

\[
O' = \begin{pmatrix} y \end{pmatrix} = \begin{pmatrix} e & f \end{pmatrix}
\]

A system is controllable if and only if the controllability matrix has full rank.

A system is observable if and only if the observability matrix has full rank.

Consider the state-space model:

\[
\dot{x} = A'x + B'u
\]

\[
y = C'x + D'u
\]

The transformed system is:

\[
\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + \begin{pmatrix} V & I \end{pmatrix} u
\]

\[
y = \begin{pmatrix} e & f \end{pmatrix} x + \begin{pmatrix} \cdot & \cdot \end{pmatrix} u
\]
The diagonal form can also be derived directly from another state-space form.

The diagonal canonical form can also be derived from another state-space form.

The diagonal form is obtained as

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where

\[
V = AVL = V
\]

Following equations are the corresponding eigenvectors of the system matrix.

The eigenvalues of the system matrix are important system properties. The eigenvalues are the same, because

\[
(V - I)\lambda = (V - I)\lambda = \lambda
\]

The eigenvalues are the same because

\[
(V - I)\lambda \neq (V - I)\lambda \neq \lambda
\]

For the transformed system, the eigenvalues are then given by

\[
(V - I)\lambda = (V - I)\lambda = \lambda
\]

The diagonal canonical form is based on an order proper system with a single input and a single output (SISO) systems. The diagonal canonical form allows at Exam allowed at Exam.

In this section, so-called canonical (i.e., standard) state-space forms for systems with a single input and a single output (SISO) are considered.
Equation (9.47) shows that \( V^T \lambda = V^T I \) is the corresponding left eigenvector of \( A \).

The vector \( \lambda \) is the eigenvalue and \( V^T \) is the corresponding left eigenvector of \( A \).

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Thus, \( V^T \) is a sparse matrix (i.e., a matrix with many zeros) with \( A \).

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In this form, the parameters of the state-space model (9.40) are observable canonical form state-space form called observable canonical form.
The latter three are maximum subdeterminants.

\[ \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \]

The matrix is the determinant of a square submatrix of maximum size.

A continuous-time system whose system matrix has the eigenvalues 

\[ \lambda \]

When the system is unstable.

In (9.46) and the subsequent derivation of (9.49), no assumptions regarding the character of the eigenvalues are made, for example, to a complex eigenvalues.

From the characteristic equation, the eigenvalues \( \lambda = \pm j \omega \), \( \omega = 0 \) when the system is unstable.

These can be written more compactly as

\[ \dot{y} = A_2 y + (0) x + f \]

When the variable transformation

\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \]

is made, the characteristic polynomial of (9.49) is

\[ \lambda^2 + 2 \lambda + 3 = 0 \]

For a MIMO system, i.e., a system with multiple input signals and multiple output signals, it is quite complicated to determine the poles from the transfer function(s) without first constructing a minimal state-space realization.

The poles are the eigenvalues of the matrix

\[ A \]

For real-valued poles, the system matrix

\[ A \]

should be non-singular, i.e., it should be of full rank. Then the eigenvalues are identical to the poles of the system. They are identical to the eigenvalues of uncontrollable and unobservable system parts.

The equivalence between eigenvalues and poles requires a controllable and observable state-space realization. This means that the equivalence between eigenvalues and poles must be considered to be possible. The result is known in the form of a transfer function (matrix) as a product of uncontrollable and unobservable system parts.

For real-valued poles, the system matrix

\[ A \]

will diverge if

\[ \det(A) = 0 \]

The transfer function of a SISO system is factorized as

\[ \frac{1}{s + \alpha} \]

The transfer function matrix

\[ \frac{1}{s + A} \]

is the determinant of a (sub)matrix of maximum size.

A continuous-time system whose system matrix

\[ A \]

has the eigenvalues

\[ \lambda \]

The latter three, are all maximum subdeterminants.

\[ \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \]

will be

\[ \det(A) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \]

The maximum subdeterminant is the determinant of a (sub)matrix of maximum size.

The following definitions are needed to determine the poles of (6.2).

The number of poles is equal to the dimension of the system. For a minimal realization, either.

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In this course, stochastic control theory is not studied.

The Kalman filter can also be used for estimation of unknown system parameters. For control of systems with stochastic disturbances, stochastic control theory can be used. The state vector is then typically estimated by a so-called filtering algorithm.

For control of systems with stochastic disturbances, stochastic control theory is an appropriate method. The state vector is then typically estimated by a so-called filtering algorithm. The output equation of a system is given by

\[ \dot{x} = A x + B u + w \]

where \( A \) is the system matrix, \( B \) is the input matrix, \( w \) is the disturbance vector, \( x \) is the state vector, and \( u \) is the control input. The output equation of the system is given by

\[ y = C x + v \]

where \( C \) is the output matrix, \( v \) is the output noise vector, and \( y \) is the output vector.

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\[ y = C x + v \]

where \( C \) is the output matrix, \( v \) is the output noise vector, and \( y \) is the output vector.
Thus, it is impossible to stabilize this system by state feedback.

As can be seen, the unstable pole \( \lambda = 1 \) cannot be reached because \( k = 0 \). Note that there are two states and one input. This means that the state feedback (9.27) does not stabilize the system, the poles are unstable. The state feedback has the same form as Example 9.4. The characteristic equation becomes

\[
0 = (\lambda - 1) (\lambda - 1) = \left[ \begin{array}{cc}
1 - \lambda & 0 \\
0 & 1 - \lambda + 1 + \lambda^2
\end{array} \right] = 0.
\]

Consider the system with \( \lambda = 1 \) and \( \lambda = 1 + k \).

Example 9.4: Non-reachable pole placement

Restrictions on pole placement

By design it is possible to choose \( k \) to give the closed-loop system the desired poles. An interesting question is whether the poles \( \lambda = 1 \) can be placed as desired.

Restrictions on pole placement

The closed-loop poles are found by solving the characteristic equation

\[
\left[ \begin{array}{cc}
\lambda - 1 & 0 \\
0 & (\lambda - 1)(\lambda - 1 - \lambda^2)
\end{array} \right] = 0
\]

The closed-loop poles are found by solving the characteristic equation

\[
(\lambda - 1)(\lambda - 1)(\lambda - 1 - \lambda^2) = 0.
\]

The characteristic equation yields

\[
(\lambda - 1)^2 + (\lambda + 1)^2 = 0.
\]

Substitution into the state equation yields

\[
\begin{align*}
(\lambda - 1)^2 & = 0 \\
(\lambda + 1)^2 & = 0.
\end{align*}
\]

Example 9.4: Stabilization by pole placement

Basic technique

Example 9.4: Pole placement

9.3.1 Pole placement

Process Dynamics and Control

9.2 State-Space Methods

9.3 Control Design by State Feedback

The control law for a continuous-time state feedback is of the form

\[
(9.3.59)
\]

The characteristic equation of the state feedback \( k = 0 \) yields

\[
(\lambda - 1)^2 = 0
\]

where \( \lambda = 1 \). In this example the control design is based on the eigenvalues of the system matrix \( K \). This is a meaningful control design because the state feedback is a continuous-time feedback. For the sake of brevity we will simplify the analysis and control design. It is often assumed that the reference signal can be obtained from the output \( x(t) \), and an external signal can be added to the output. In the case where \( K \) is a matrix of controller gains and \( n_1 \) is an external signal for the state the characteristic equation of the state feedback is

\[
(9.3.59)
\]

The control law for a continuous-time state feedback is of the form

\[
(9.3.59)
\]
Poles far from the origin mean fast dynamics, but also large input changes. For systems of "high" order, the dominant dynamics are determined by the undamped natural frequency of the system, \( \omega_n \), and the constant of proportionality of the system, \( \zeta \). Complex poles result in a step response with overshoot and/or oscillations. The distance of the poles from the origin should generally be in the area close to the origin for the system to behave as desired.

The control law has the form:

\[
L(s) = K(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

where \( \omega_n \) is the undamped natural frequency of the system, \( \zeta \) is the damping ratio, and \( K \) is the gain. This control law results in a response with overshoot and oscillations, but the poles are placed in certain regions of the left-half plane.

Example 9.6: Step response specifications for an underdamped system.

The motor is to be controlled by state feedback to yield a closed-loop system with the characteristic polynomial:

\[
(s^2 + 2\zeta\omega_n s + \omega_n^2)(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \omega_n^2)
\]

The poles of the second-order system are on the real axis and are complex-conjugated. The system is stable and the poles should be placed in the desired area for a good performance.

Laboratory 9.3: Control process.

Laboratory 9.4: Control process.
Process Control Laboratory

Substitution into the state-equation yields

\[
\begin{align*}
\dot{x} &= A x + B u \\
\end{align*}
\]

The transfer function for the closed-loop system becomes

\[
G(s) = \frac{C(s)}{A(s)} = \frac{C(s)}{A(s)}
\]

The specifications mean that the controller parameters should be selected as

\[
\begin{align*}
K_1 &= 1 \\
K_2 &= 1 \\
\end{align*}
\]

Step-response specifications yield

\[
\begin{align*}
\begin{bmatrix} y_1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x \\
\end{align*}
\]