Fourier series

Consider a discrete-time signal

\[ \{ x(n) \} = \{ x(0), x(1), \ldots, x(N - 1) \} \]

We can expand it in terms of sinusoidals as

\[ x(n) = \sum_{k=1}^{N/2} A_k \cos(2\pi f_k n + \phi_k), \quad n = 0, 1, \ldots, N - 1 \]

For given frequencies \( f_k \), the \( N \) parameters \( A_k, \phi_k, k = 1, \ldots, N \) can be found by solving the equation system composed of the above equations.

This equation system is, however, nonlinear in \( A_k \) and \( \phi_k \)!
A linear system of equations is obtained by observing that

\[ A \cos(2\pi fn + \phi) = a \cos(2\pi fn) + b \sin(2\pi fn) \]

where \( a = A \cos(\phi) \), \( b = -A \sin(\phi) \)

[ Follows from trigonometric identity:

\[ \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \]

\[ \Rightarrow \]

\[ \cos(2\pi fn + \phi) = \cos(\phi) \cos(2\pi fn) - \sin(\phi) \sin(2\pi fn) \]
Selection of frequencies $f_k$:

Observe that

$$\cos(2\pi fn + \phi) = \cos(-2\pi fn - \phi)$$

$$= \cos(-2\pi fn - \phi + 2\pi n)$$

$$= \cos(2\pi (1 - f)n - \phi)$$

⇒ the frequencies $f$ and $1 - f$ result in the same signal, if the sign of the phase is inverted.

⇒ It is sufficient to consider frequencies $0 \leq f \leq 1/2$.

Taking equidistant frequencies gives

$$f_k = 0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{1}{2} (= \frac{N/2}{N}) \quad (N \text{ even})$$

$$f_k = 0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{1 - 1/N}{2} \quad (= \frac{(N - 1)/2}{N}) \quad (N \text{ odd})$$
Hence we have the Fourier series:

\[ x(n) = \sum_{k=0}^{M} a_k \cos\left(2\pi \frac{k}{N} n\right) + b_k \sin\left(2\pi \frac{k}{N} n\right), \quad n = 0, 1, \ldots, N - 1 \]

where \( M = N/2 \), for even \( N \), and \( M = (N - 1)/2 \), for odd \( N \).

Notice, that for

- \( k = 0 \): \( \cos(0 \cdot n) = 1 \) and \( \sin(0 \cdot n) = 0 \), and for

- \( k = N/2 \): \( \cos(2\pi \frac{1}{2} n) = (-1)^n \) and \( \sin(2\pi \frac{1}{2} n) = 0 \).

\[ \Rightarrow b_0 = 0 \text{ and } b_{N/2} = 0 \]
It follows that the Fourier series is determined by the $N$ coefficients

$$a_0, a_1, \ldots, a_{N/2}, b_1, \ldots, b_{(N-1)/2} \quad \text{(even } N\text{)}$$

$$a_0, a_1, \ldots, a_{(N-1)/2}, b_1, \ldots, b_{(N-1)/2} \quad \text{(odd } N\text{)}$$

The $N$ coefficients can be determined from the $N$ relations for $x(0), x(1), \ldots, x(N-1)$.

This an $N$-dimensional linear equation system.

The solution can be found efficiently by exploiting properties of the trigonometric functions.
SOLUTION: Multiply $x(n)$ with $\cos(2\pi \frac{l}{N} n)$ and sum over $n$:

$$\sum_{n=0}^{N-1} x(n) \cos(2\pi \frac{l}{N} n) = \sum_{k=0}^{M} \sum_{n=0}^{N-1} a_k \cos(2\pi \frac{l}{N} n) \cos(2\pi \frac{k}{N} n) + \sum_{k=0}^{M} \sum_{n=0}^{N-1} b_k \cos(2\pi \frac{l}{N} n) \sin(2\pi \frac{k}{N} n)$$

Use the fact that

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{l}{N} n) \cos(2\pi \frac{k}{N} n) = \begin{cases} N/2 & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases}$$

$$\sum_{n=0}^{N-1} \cos(2\pi \frac{l}{N} n) \sin(2\pi \frac{k}{N} n) = 0, \text{ all } l, k$$
\[ \Rightarrow \quad \sum_{n=0}^{N-1} x(n) \cos(2\pi \frac{l}{N} n) = \frac{1}{2} Na_l \]

which gives \( a_l \).

Multiplying \( x(n) \) with \( \sin(2\pi \frac{l}{N} n) \) and summing over \( n \) gives analogously:

\[ \sum_{n=0}^{N-1} x(n) \sin(2\pi \frac{l}{N} n) = \frac{1}{2} Nb_l \]

Hence we have:

\[ a_l = \frac{2}{N} \sum_{n=0}^{N-1} x(n) \cos(2\pi \frac{l}{N} n) \]

\[ b_l = \frac{2}{N} \sum_{n=0}^{N-1} x(n) \sin(2\pi \frac{l}{N} n) \]
Proof of

\[ \sum_{n=0}^{N-1} \cos(2\pi \frac{l}{N}n) \cos(2\pi \frac{k}{N}n) = \begin{cases} N/2 & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \]

Use the formula \( \cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta) \):

\[ \cos(2\pi \frac{l}{N}n) \cos(2\pi \frac{k}{N}n) = \frac{1}{2} \cos(2\pi \frac{l-k}{N}n) + \frac{1}{2} \cos(2\pi \frac{l+k}{N}n) \]

and

\[ \sum_{n=0}^{N-1} \cos(2\pi \frac{m}{N}n) = \begin{cases} 0 & \text{if } m \neq 0 \\ N & \text{if } m = 0 \end{cases} \]

\[ \sum_{n=0}^{N-1} \cos(2\pi \frac{l}{N}n) \sin(2\pi \frac{k}{N}n) = 0 \] can be shown similarly
FOURIER SERIES OF SAMPLED SIGNAL

Consider a continuous-time signal $x(t)$ which is sampled with sampling interval $T_s$ to give sampled signal

$$\{x(nT_s)\} = \{x(0), x(T_s), \ldots, x((N - 1)T_s)\}$$

Now the time instants are $nT_s$, and the sampling frequency is $f_s = 1/T_s$

$\Rightarrow$ Fourier series takes the form

$$x(nT_s) = \sum_{k=0}^{M} a_k \cos(2\pi \frac{k f_s}{N} nT_s) + b_k \sin(2\pi \frac{k f_s}{N} nT_s)$$

$$= \sum_{k=0}^{M} a_k \cos(2\pi \frac{k}{N} n) + b_k \sin(2\pi \frac{k}{N} n), \quad n = 0, 1, \ldots, N - 1$$

i.e., it has the same form as before.